Functions and Equations workbook
~ categorized past IB Paper 1 and Paper 2 examination questions ~

IB DP Mathematics Standard Level
Topic 2
This workbook contains past Paper 1 and Paper 2 IB examination questions categorized according to major concepts in this topic.

Contents

- **2.1 Composite and Inverse Functions**
  - Paper 1 Questions
    - 24 questions; 142 marks
  - Paper 2 Questions
    - 4 questions; 24 marks

- **2.2 Graphs of Functions**
  - Paper 1 Questions
    - 2 questions; 12 marks
  - Paper 2 Questions
    - 9 questions; 50 marks

- **2.3 Transformation of Graphs**
  - Paper 1 Questions
    - 15 questions 105 marks
  - Paper 2 Questions
    - 3 questions; 26 marks

- **2.4 Quadratic Functions**
  - Paper 1 Questions
    - 24 questions 130 marks
  - Paper 2 Questions
    - 6 questions; 55 marks

- **2.5 Logarithmic and Exponential Functions**
  - Paper 1 Questions
    - 5 questions 29 marks
  - Paper 2 Questions
    - 17 questions; 136 marks

- **2.6 Solving Equations**
  - Paper 1 Questions
    - 4 questions 30 marks
  - Paper 2 Questions
    - 9 questions; 71 marks

*The use of GDC is not permitted for Paper 1 but is required for Paper 2 questions.*
2.1 Composite and Inverse Functions

1. Let \( f(x) = 7 - 2x \) and \( g(x) = x + 3 \).
   (a) Find \((g \circ f)(x)\).   
   \[ (2) \]
   (b) Write down \( g^{-1}(x) \).   
   \[ (1) \]
   (c) Find \((f \circ g^{-1})(5)\).   
   \[ (2) \]
   (Total 5 marks)

2. Let \( f(x) = \log_3 \sqrt{x} \), for \( x > 0 \).
   (a) Show that \( f^{-1}(x) = 3^{2x} \).   
   \[ (2) \]
   (b) Write down the range of \( f^{-1} \).   
   \[ (1) \]
   (c) Let \( g(x) = \log_3 x \), for \( x > 0 \). Find the value of \((f^{-1} \circ g)(2)\), giving your answer as an integer.   
   \[ (4) \]
   (Total 7 marks)
3. Let \( f(x) = \cos 2x \) and \( g(x) = 2x^2 - 1 \).

(a) Find \( f\left(\frac{\pi}{2}\right) \).

(b) Find \( (g \circ f)\left(\frac{\pi}{2}\right) \).

(c) Given that \( (g \circ f)(x) \) can be written as \( \cos(kx) \), find the value of \( k, k \in \mathbb{Z} \).

(Total 7 marks)

4. Let \( f(x) = x^2 \) and \( g(x) = 2x - 3 \).

(a) Find \( g^{-1}(x) \).

(b) Find \( (f \circ g)(4) \).

(Total 5 marks)
5. Let \( f(x) = 2x^3 + 3 \) and \( g(x) = e^{3x} - 2 \).

(a) (i) Find \( g(0) \).

(ii) Find \((f \circ g)(0)\).

(b) Find \( f^{-1}(x) \).

(Total 8 marks)

6. Let \( f(x) = \ln(x + 5) + \ln 2 \), for \( x > -5 \).

(a) Find \( f^{-1}(x) \).

(b) Let \( g(x) = e^x \). Find \((g \circ f)(x)\), giving your answer in the form \( ax + b \), where \( a, b, \in \mathbb{Z} \).

(Total 7 marks)
7. Consider \( f(x) = \sqrt{x - 5} \).

(a) Find \( f(11) \), \( f(86) \), and \( f(5) \).

(b) Find the values of \( x \) for which \( f \) is undefined.

(c) Let \( g(x) = x^2 \). Find \( (g \circ f)(x) \).

(Total 7 marks)

8. Let \( f(x) = \sqrt{x+4}, x \geq -4 \) and \( g(x) = x^2, x \in \mathbb{R} \).

(a) Find \( (g \circ f) \) (3).

(b) Find \( f^{-1}(x) \).

(c) Write down the domain of \( f^{-1} \).

(Total 6 marks)
9. Let \( f(x) = x^3 - 4 \) and \( g(x) = 2x \).
   
   (a) Find \((g \circ f)(-2)\).
   
   (b) Find \(f^{-1}(x)\).

   (Total 6 marks)

10. Let \( g(x) = 3x - 2 \), \( h(x) = \frac{5x}{x-4}, x \neq 4 \).

   (a) Find an expression for \((h \circ g)(x)\). Simplify your answer.

   (b) Solve the equation \((h \circ g)(x) = 0\).

   (Total 6 marks)
11. The functions $f$ and $g$ are defined by $f: \mapsto 3x$, $g: x \mapsto x + 2$.

(a) Find an expression for $(f \circ g)(x)$.

(b) Show that $f^{-1}(18) + g^{-1}(18) = 22$.

(Total 6 marks)

12. Consider the functions $f(x) = 2x$ and $g(x) = \frac{1}{x-3}, x \neq 3$.

(a) Calculate $(f \circ g)(4)$.

(b) Find $g^{-1}(x)$.

(c) Write down the domain of $g^{-1}$.

(Total 6 marks)
13. Let \( f(x) = 2x + 1 \) and \( g(x) = 3x^2 - 4 \). Find

(a) \( f^{-1}(x) \);
(b) \( (g \circ f)(-2) \);
(c) \( (f \circ g)(x) \).

(Total 6 marks)

14. Let \( f(x) = e^{-x} \), and \( g(x) = \frac{x}{1+x}, \ x \neq -1 \). Find

(a) \( f^{-1}(x) \);
(b) \( (g \circ f)(x) \).

(Total 6 marks)
15. Let \( f(x) = 2^x \), and \( g(x) = \frac{x}{x-2}, \ (x \neq 2) \). Find

(a) \((g \circ f)(3)\);
(b) \(g^{-1}(5)\).

(Total 6 marks)

16. Consider the functions \( f : x \mapsto 4(x - 1) \) and \( g : x \mapsto \frac{6-x}{2} \).

(a) Find \( g^{-1} \).
(b) Solve the equation \((f \circ g^{-1})(x) = 4\).

(Total 6 marks)
17. Two functions $f, g$ are defined as follows:

\[ f : x \to 3x + 5 \]
\[ g : x \to 2(1 - x) \]

Find

(a) $f^{-1}(2)$;

(b) $(g \circ f)(-4)$.

(Total 4 marks)

18. Two functions $f$ and $g$ are defined as follows:

\[ f(x) = \cos x, \quad 0 \leq x \leq 2\pi; \]
\[ g(x) = 2x + 1, \quad x \in \mathbb{R}. \]

Solve the equation $(g \circ f)(x) = 0$.

(Total 4 marks)
19. Let \( f(x) = \sqrt{x} \), and \( g(x) = 2^x \). Solve the equation \((f^{-1} \circ g)(x) = 0.25\).
(Total 4 marks)

20. Let \( f(x) = e^{x+3} \).

(a) (i) Show that \( f^{-1}(x) = \ln x - 3 \).
(ii) Write down the domain of \( f^{-1} \).

(b) Solve the equation \( f^{-1}(x) = \ln \left( \frac{1}{x} \right) \).

(Total 7 marks)
21. Let \( f(x) = k \log_2 x \).

(a) Given that \( f^{-1}(1) = 8 \), find the value of \( k \).

(b) Find \( f^{-1}\left(\frac{2}{3}\right) \).

22. The function \( f \) is given by \( f(x) = e^{(x-11)} - 8 \).

(a) Find \( f^{-1}(x) \).

(b) Write down the domain of \( f^{-1}(x) \).
23. The function \( f \) is given by \( f(x) = x^2 - 6x + 13 \), for \( x \geq 3 \).

(a) Write \( f(x) \) in the form \((x - a)^2 + b\).

(b) Find the inverse function \( f^{-1} \).

(c) State the domain of \( f^{-1} \).

(Total 6 marks)

24. Given that \( f(x) = 2e^{3x} \), find the inverse function \( f^{-1}(x) \).

(Total 4 marks)
1. Let $f$ be the function given by $f(x) = e^{0.5x}$, $0 \leq x \leq 3.5$. The diagram shows the graph of $f$.

(a) On the same diagram, sketch the graph of $f^{-1}$. 

(b) Write down the range of $f^{-1}$. 

(c) Find $f^{-1}(x)$. 

(Total 7 marks)

2. Let $f(x) = 3x$, $g(x) = 2x - 5$ and $h(x) = (f \circ g)(x)$.

(a) Find $h(x)$. 

(b) Find $h^{-1}(x)$. 

(Total 5 marks)
3. Let \( f(x) = \frac{3x}{2} + 1 \), \( g(x) = 4\cos\left(\frac{x}{3}\right) - 1 \). Let \( h(x) = (g \circ f)(x) \).

(a) Find an expression for \( h(x) \).  
(3)

(b) Write down the period of \( h \).  
(1)

(c) Write down the range of \( h \).  
(2)

(Total 6 marks)

4. The functions \( f \) and \( g \) are defined by \( f : x \mapsto 3x \), \( g : x \mapsto x + 2 \).

(a) Find an expression for \( (f \circ g)(x) \).  
(2)

(b) Find \( f^{-1}(18) + g^{-1}(18) \).  
(4)

(Total 6 marks)
2.2 Graphs of Functions

1. Consider the line $L$ with equation $y + 2x = 3$. The line $L_1$ is parallel to $L$ and passes through the point $(6, -4)$.
   
   (a) Find the gradient of $L_1$.
   
   (b) Find the equation of $L_1$ in the form $y = mx + b$.
   
   (c) Find the $x$-coordinate of the point where line $L_1$ crosses the $x$-axis.

   (Total 6 marks)

2. Consider the following relations between two variables $x$ and $y$.

   A. $y = \sin x$
   
   B. $y$ is directly proportional to $x$
   
   C. $y = 1 + \tan x$
   
   D. speed $y$ as a function of time $x$, constant acceleration
   
   E. $y = 2^x$
   
   F. distance $y$ as a function of time $x$, velocity decreasing

   Each sketch on the left could represent exactly two of the above relations on a certain interval. Complete the table below, by writing the letter for the two relations that each sketch could represent.

   (Total 6 marks)
2.2 Graphs of Functions

1. Let \( f(x) = 3 \sin 2x \), for \( 1 \leq x \leq 4 \) and \( g(x) = -5x^2 + 27x - 35 \) for \( 1 \leq x \leq 4 \). The graph of \( f \) is shown below.
   
   (a) On the same diagram, sketch the graph of \( g \).
   
   (b) One solution of \( f(x) = g(x) \) is 1.89. Write down the other solution.
   
   (c) Let \( h(x) = g(x) - f(x) \). Given that \( h(x) > 0 \) for \( p < x < q \), write down the value of \( p \) and of \( q \).

   (Total 6 marks)

2. The function \( f \) is defined by \( f(x) = \frac{3}{\sqrt{9-x^2}} \), for \(-3 < x < 3\).
   
   (a) On the grid below, sketch the graph of \( f \).
   
   (b) Write down the equation of each vertical asymptote.
   
   (c) Write down the range of the function \( f \).

   (Total 6 marks)
3. Let \( f(x) = 2 + \cos(2x) - 2\sin(0.5x) \) for \( 0 \leq x \leq 3 \), where \( x \) is in radians.

(a) On the grid below, sketch the curve of \( y = f(x) \), indicating clearly the point \( P \) on the curve where the derivative is zero.

(b) Write down the solutions of \( f(x) = 0 \).

(Total 6 marks)

4. (a) On the following diagram, sketch the graphs of \( y = e^x \) and \( y = \cos x \) for \(-2 \leq x \leq 1\).

(b) The equation \( e^x = \cos x \) has a solution between \(-2\) and \(-1\). Find this solution.

(Total 4 marks)
5. The diagram below shows the graph of \( y = x \sin \left( \frac{x}{3} \right) \), for \( 0 \leq x < m \), and \( 0 \leq y < n \), where \( x \) is in radians and \( m \) and \( n \) are integers. Find the value of

(a) \( m \);

(b) \( n \).

(Total 4 marks)

6. \( f(x) = 4 \sin \left( \frac{3x + \pi}{2} \right) \). For what values of \( k \) will the equation \( f(x) = k \) have no solutions?

(Total 4 marks)
7. Consider \( f(x) = 2 - x^2 \), for \(-2 \leq x \leq 2\) and \( g(x) = \sin e^x \), for \(-2 \leq x \leq 2\). The graph of \( f \) is given below.

(a) On the diagram above, sketch the graph of \( g \).  

(b) Solve \( f(x) = g(x) \). 

(c) Write down the set of values of \( x \) such that \( f(x) > g(x) \). 

(Total 7 marks)

8. Consider the function \( f(x) = px^3 + qx^2 + rx \). Part of the graph of \( f \) is shown below. The graph passes through the origin \( O \) and the points \( A(-2, -8) \), \( B(1, -2) \) and \( C(2, 0) \).

(a) Find three linear equations in \( p \), \( q \) and \( r \).  

(b) Hence find the value of \( p \), of \( q \) and of \( r \). 

(Total 7 marks)
9. The function \( f \) is defined by \( f(x) = \frac{3}{\sqrt{9-x^2}} \), for \(-3 < x < 3\).

(a) On the grid below, sketch the graph of \( f \).

(b) Write down the equation of each vertical asymptote.

(c) Write down the range of the function \( f \).

(Total 6 marks)
1. Let \( f(x) = x^2 + 4 \) and \( g(x) = x - 1 \).

(a) Find \((f \circ g)(x)\).

The vector \( \begin{pmatrix} 3 \\ -1 \end{pmatrix} \) translates the graph of \((f \circ g)\) to the graph of \(h\).

(b) Find the coordinates of the vertex of the graph of \(h\).

(c) Show that \( h(x) = x^2 - 8x + 19 \).

(d) The line \( y = 2x - 6 \) is a tangent to the graph of \(h\) at the point \(P\). Find the \(x\)-coordinate of \(P\).

(Total 12 marks)
2. The diagram below shows the graph of a function \( f(x) \), for \(-2 \leq x \leq 4\).

(a) Let \( h(x) = f(-x) \). Sketch the graph of \( h \) on the grid below.

(b) Let \( g(x) = \frac{1}{2} f(x - 1) \). The point \( A(3, 2) \) on the graph of \( f \) is transformed to the point \( P \) on the graph of \( g \). Find the coordinates of \( P \).

(Total 5 marks)
3. Let \( f(x) = x^2 \) and \( g(x) = 2(x - 1)^2 \).

(a) The graph of \( g \) can be obtained from the graph of \( f \) using two transformations. Give a full geometric description of each of the two transformations. 

(b) The graph of \( g \) is translated by the vector \( \begin{pmatrix} 3 \\ -2 \end{pmatrix} \) to give the graph of \( h \). The point \((-1, 1)\) on the graph of \( f \) is translated to the point \( P \) on the graph of \( h \). Find the coordinates of \( P \).

(Total 6 marks)

4. Part of the graph of a function \( f \) is shown in the diagram below.

(a) On the same diagram sketch the graph of \( y = -f(x) \).

(b) Let \( g(x) = f(x + 3) \).

(i) Find \( g(-3) \).

(ii) Describe fully the transformation that maps the graph of \( f \) to the graph of \( g \).

(Total 6 marks)
5. Let \( f(t) = a \cos b(t - c) + d \), \( t \geq 0 \). Part of the graph of \( y = f(t) \) is given below. When \( t = 3 \), there is a maximum value of 29, at M. When \( t = 9 \), there is a minimum value of 15.

(a) (i) Find the value of \( a \).

(ii) Show that \( b = \frac{\pi}{6} \).

(iii) Find the value of \( d \).

(iv) Write down a value for \( c \).

(b) Let \( M' \) be the image of \( M \) under \( P \). Find the coordinates of \( M' \).

The transformation \( P \) is given by a horizontal stretch of a scale factor of \( \frac{1}{2} \), followed by a translation of \( \left( \begin{array}{c} 3 \\ -10 \end{array} \right) \).

(c) Find \( g(t) \) in the form \( g(t) = 7 \cos B(t - C) + D \).

(d) Give a full geometric description of the transformation that maps the graph of \( g \) to the graph of \( f \).

(Total 16 marks)
6. Let \( f(x) = \frac{1}{x}, x \neq 0 \).

(a) Sketch the graph of \( f \).

(b) Find an expression for \( g(x) \).

(c) (i) Find the intercepts of \( g \).
(ii) Write down the equations of the asymptotes of \( g \).
(iii) Sketch the graph of \( g \).
7. The graph of a function \( f \) is shown in the diagram below. The point \( A (-1, 1) \) is on the graph, and \( y = -1 \) is a horizontal asymptote.

(a) Let \( g(x) = f(x - 1) + 2 \). On the diagram, sketch the graph of \( g \).

(b) Write down the equation of the horizontal asymptote of \( g \).

(c) Let \( A' \) be the point on the graph of \( g \) corresponding to point \( A \). Write down the coordinates of \( A' \).

(Total 6 marks)
8. The following diagram shows part of the graph of \( f(x) \).

Consider the five graphs in the diagrams labelled A, B, C, D, E below.

(a) Which diagram is the graph of \( f(x + 2) \)?

(b) Which diagram is the graph of \( -f(x) \)?

(c) Which diagram is the graph of \( f(-x) \)

(Total 6 marks)
9. The graph of \( y = f(x) \) is shown in the diagram.

(a) On each of the following diagrams draw the required graph,
   
   (i) \( y = 2f(x) \);

   (ii) \( y = f(x - 3) \).

(b) The point \( A(3, -1) \) is on the graph of \( f \). The point \( A' \) is the corresponding point on the graph of \( y = -f(x) + 1 \). Find the coordinates of \( A' \).

(Total 6 marks)
The sketch shows part of the graph of \( y = f(x) \) which passes through the points A(–1, 3), B(0, 2), C(1, 0), D(2, 1) and E(3, 5).

A second function is defined by \( g(x) = 2f(x - 1) \).

(a) Calculate \( g(0), g(1), g(2) \) and \( g(3) \).

(b) On the same axes, sketch the graph of the function \( g(x) \).

(Total 6 marks)
11. The following diagram shows the graph of $y = f(x)$. It has minimum and maximum points at $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$.

(a) On the same diagram, draw the graph of $y = f(x - 1) + \frac{3}{2}$.

(b) What are the coordinates of the minimum and maximum points of $y = f(x - 1) + \frac{3}{2}$?

(Total 4 marks)
12. The diagram shows the graph of \( y = f(x) \), with the \( x \)-axis as an asymptote.

(a) On the same axes, draw the graph of \( y = f(x + 2) - 3 \), indicating the coordinates of the images of the points A and B.

(b) Write down the equation of the asymptote to the graph of \( y = f(x + 2) - 3 \).

(Total 4 marks)

13. The diagrams show how the graph of \( y = x^2 \) is transformed to the graph of \( y = f(x) \) in three steps. For each diagram give the equation of the curve.

(Total 4 marks)
14. Three of the following diagrams I, II, III, IV represent the graphs of

(a) \( y = 3 + \cos 2x \)

(b) \( y = 3 \cos (x + 2) \)

(c) \( y = 2 \cos x + 3. \)

Identify which diagram represents which graph.
15. Consider the graph of \(f\) shown below.

(a) On the same grid sketch the graph of \(y = f(-x)\).

The following four diagrams show images of \(f\) under different transformations.

(b) Complete the following table.

<table>
<thead>
<tr>
<th>Description of transformation</th>
<th>Diagram letter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal stretch with scale factor 1.5</td>
<td></td>
</tr>
<tr>
<td>Maps (f) to (f(x) + 1)</td>
<td></td>
</tr>
</tbody>
</table>

(2)

(c) Give a full geometric description of the transformation that gives the image in Diagram A.

(2)

(Total 6 marks)
1. Let $f(x) = \frac{1}{3}x^3 - x^2 - 3x$. Part of the graph of $f$ is shown below. There is a maximum point at A and a minimum point at B(3, -9).

(a) Find the coordinates of A.  

(b) Write down the coordinates of

(i) the image of B after reflection in the $y$-axis;

(ii) the image of B after translation by the vector $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$;

(iii) the image of B after reflection in the $x$-axis followed by a horizontal stretch with scale factor $\frac{1}{2}$.

(Total 14 marks)
2. (a) The diagram shows part of the graph of the function \( f(x) = \frac{q}{x - p} \). The curve passes through the point A (3, 10). The line (CD) is an asymptote. Find the value of \( p \) and \( q \).

(b) The graph of \( f(x) \) is transformed as shown in the following diagram. The point A is transformed to \( A' (3, -10) \).

Give a full geometric description of the transformation.

(Total 6 marks)
3. Let \( f(x) = 2x + 1 \).

(a) On the grid below draw the graph of \( f(x) \) for \( 0 \leq x \leq 2 \).

(b) Let \( g(x) = f(x + 3) - 2 \). On the grid below draw the graph of \( g(x) \) for \( -3 \leq x \leq -1 \).

(Total 6 marks)
2.4 Quadratic Functions

1. Consider \( f(x) = 2kx^2 - 4kx + 1 \), for \( k \neq 0 \). The equation \( f(x) = 0 \) has two equal roots.

   (a) Find the value of \( k \).

   (b) The line \( y = p \) intersects the graph of \( f \). Find all possible values of \( p \).

   (Total 7 marks)

2. Let \( f(x) = 8x - 2x^2 \). Part of the graph of \( f \) is shown below.

   (a) Find the \( x \)-intercepts of the graph.

   (b) (i) Write down the equation of the axis of symmetry.

   (ii) Find the \( y \)-coordinate of the vertex.

   (Total 7 marks)
3. Let \( f(x) = p(x - q)(x - r) \). Part of the graph of \( f \) is shown below. The graph passes through the points \((-2, 0), (0, -4)\) and \((4, 0)\).

(a) Write down the value of \( q \) and of \( r \). (2)

(b) Write down the equation of the axis of symmetry. (1)

(c) Find the value of \( p \). (3)

(Total 6 marks)

4. The following diagram shows part of the graph of \( f \), where \( f(x) = x^2 - x - 2 \).

(a) Find both \( x \)-intercepts. (4)

(b) Find the \( x \)-coordinate of the vertex. (2)

(Total 6 marks)
5. The quadratic function $f$ is defined by $f(x) = 3x^2 - 12x + 11$.

(a) Write $f$ in the form $f(x) = 3(x - h)^2 - k$.  

(b) The graph of $f$ is translated 3 units in the positive $x$-direction and 5 units in the positive $y$-direction. Find the function $g$ for the translated graph, giving your answer in the form $g(x) = 3(x - p)^2 + q$.  

(Total 6 marks)

6. Consider two different quadratic functions of the form $f(x) = 4x^2 - qx + 25$. The graph of each function has its vertex on the $x$-axis.

(a) Find both values of $q$.

(b) For the greater value of $q$, solve $f(x) = 0$.

(c) Find the coordinates of the point of intersection of the two graphs.  

(Total 6 marks)
7. The following diagram shows part of the graph of \( f(x) = 5 - x^2 \) with vertex \( V(0, 5) \). Its image \( y = g(x) \) after a translation with vector \( \begin{pmatrix} h \\ k \end{pmatrix} \) has vertex \( T(3, 6) \).

(a) Write down the value of \( h \) and \( k \).

(b) Write down an expression for \( g(x) \).

(c) On the same diagram, sketch the graph of \( y = g(-x) \).

(Total 6 marks)

8. (a) Express \( y = 2x^2 - 12x + 23 \) in the form \( y = 2(x - c)^2 + d \).

The graph of \( y = x^2 \) is transformed into the graph of \( y = 2x^2 - 12x + 23 \) by the transformations

- a vertical stretch with scale factor \( k \) followed by
- a horizontal translation of \( p \) units followed by
- a vertical translation of \( q \) units.

(b) Write down the value of \( k \), \( p \), and \( q \).

(Total 6 marks)
9. Part of the graph of the function \( y = d (x - m)^2 + p \) is given in the diagram below. The x-intercepts are (1, 0) and (5, 0). The vertex is \( V(m, 2) \).

(a) Write down the value of \( m \) and \( p \).

(b) Find \( d \).

(Total 6 marks)

10. The quadratic function \( f \) is defined by \( f(x) = 3x^2 - 12x + 11 \).

(a) Write \( f \) in the form \( f(x) = 3(x - h)^2 - k \).

(b) The graph of \( f \) is translated 3 units in the positive \( x \)-direction and 5 units in the positive \( y \)-direction. Find the function \( g \) for the translated graph, giving your answer in the form \( g(x) = 3(x - p)^2 + q \).

(Total 6 marks)
11. Part of the graph of \( f(x) = (x - p)(x - q) \) is shown below. The vertex is at C. The graph crosses the y-axis at B.

(a) Write down the value of \( p \) and of \( q \).

(b) Find the coordinates of C.

(c) Write down the y-coordinate of B.

(Total 6 marks)

12. The equation \( x^2 - 2kx + 1 = 0 \) has two distinct real roots. Find the set of all possible values of \( k \).

(Total 6 marks)
13. The equation \( kx^2 + 3x + 1 = 0 \) has exactly one solution. Find the value of \( k \).

(Total 6 marks)

14. The diagram shows part of the graph of \( y = a (x - h)^2 + k \). The graph has its vertex at \( P \), and passes through the point \( A \) with coordinates \((1, 0)\).

(a) Write down the value of \( h \) and \( k \).

(b) Calculate the value of \( a \).

(Total 6 marks)
15. Consider the function $f(x) = 2x^2 - 8x + 5$.

(a) Express $f(x)$ in the form $a(x - p)^2 + q$, where $a, p, q \in \mathbb{Z}$.

(b) Find the minimum value of $f(x)$.

(Total 6 marks)

16. Consider the function $f(x) = 2x^2 - 8x + 5$.

(a) Express $f(x)$ in the form $a(x - p)^2 + q$, where $a, p, q \in \mathbb{Z}$.

(b) Find the minimum value of $f(x)$.

(Total 6 marks)
17. The diagram shows part of the graph with equation \( y = x^2 + px + q \). The graph cuts the \( x \)-axis at \(-2\) and \(3\). Find the value of \( p \) and \( q \).

(Total 4 marks)

18. The diagram shows parts of the graphs of \( y = x^2 \) and \( y = 5 - 3(x - 4)^2 \). The graph of \( y = x^2 \) may be transformed into the graph of \( y = 5 - 3(x - 4)^2 \) by these transformations.

A reflection in the line \( y = 0 \) followed by a vertical stretch with scale factor \( k \) followed by a horizontal translation of \( p \) units followed by a vertical translation of \( q \) units.

Write down the value of \( k \), \( p \), and \( q \).

(Total 4 marks)
19. (a) Express \( f(x) = x^2 - 6x + 14 \) in the form \( f(x) = (x - h)^2 + k \), where \( h \) and \( k \) are to be determined.

(b) Hence, or otherwise, write down the coordinates of the vertex of the parabola with equation \( y = x^2 - 6x + 14 \).

(Total 4 marks)

20. The quadratic equation \( 4x^2 + 4kx + 9 = 0 \), \( k > 0 \) has exactly one solution for \( x \). Find the value of \( k \).

(Total 4 marks)
21. The diagram shows the graph of the function \( y = ax^2 + bx + c \).

Complete the table below to show whether each expression is positive, negative or zero.

<table>
<thead>
<tr>
<th>Expression</th>
<th>positive</th>
<th>negative</th>
<th>zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( b^2 - 4ac )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( b )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(Total 4 marks)

22. (a) Factorize \( x^2 - 3x - 10 \).

(b) Solve the equation \( x^2 - 3x - 10 = 0 \).

(Total 4 marks)
23. The diagram represents the graph of the function \( f: x \mapsto (x - p)(x - q) \).

(a) Write down the values of \( p \) and \( q \).

(b) The function has a minimum value at the point \( C \). Find the \( x \)-coordinate of \( C \).

(Total 4 marks)

24. The diagram shows the parabola \( y = (7 - x)(1 + x) \). The points \( A \) and \( C \) are the \( x \)-intercepts and the point \( B \) is the maximum point. Find the coordinates of \( A \), \( B \) and \( C \).

(Total 4 marks)
2.4 Quadratic Functions

1. Let \( f(x) = 2x^2 - 12x + 5 \).

   (a) Express \( f(x) \) in the form \( f(x) = 2(x - h)^2 - k \).  
   \( \quad \text{(3 marks)} \)

   (b) Write down the vertex of the graph of \( f \).  
   \( \quad \text{(2 marks)} \)

   (c) Write down the equation of the axis of symmetry of the graph of \( f \).  
   \( \quad \text{(1 mark)} \)

   (d) Find the \( y \)-intercept of the graph of \( f \).  
   \( \quad \text{(2 marks)} \)

   (e) The \( x \)-intercepts of \( f \) can be written as \( \frac{p \pm \sqrt{q}}{r} \), where \( p, q, r \in \mathbb{Z} \). Find the value of \( p \), of \( q \), and of \( r \).  
   \( \quad \text{(7 marks)} \)

   (Total 15 marks)
2. Let \( f(x) = 3(x + 1)^2 - 12 \).

(a) Show that \( f(x) = 3x^2 + 6x - 9 \).

(b) For the graph of \( f \)

(i) write down the coordinates of the vertex;

(ii) write down the equation of the axis of symmetry;

(iii) write down the \( y \)-intercept;

(iv) find both \( x \)-intercepts.

(c) Hence sketch the graph of \( f \).

(d) Let \( g(x) = x^2 \). The graph of \( f \) may be obtained from the graph of \( g \) by the two transformations:

a stretch of scale factor \( t \) in the \( y \)-direction

followed by

a translation of \( \left( \begin{array}{c} p \\ q \end{array} \right) \)

Find \( \left( \begin{array}{c} p \\ q \end{array} \right) \) and the value of \( t \).
3. Let \( f(x) = a(x - 4)^2 + 8 \).
   (a) Write down the coordinates of the vertex of the curve of \( f \).
   (b) Given that \( f(7) = -10 \), find the value of \( a \).
   (c) Hence find the \( y \)-intercept of the curve of \( f \).

(Total 6 marks)

4. Let \( f(x) = 3x^2 \). The graph of \( f \) is translated 1 unit to the right and 2 units down. The graph of \( g \) is the image of the graph of \( f \) after this translation.
   (a) Write down the coordinates of the vertex of the graph of \( g \).
   (b) Express \( g \) in the form \( g(x) = 3(x - p)^2 + q \).

The graph of \( h \) is the reflection of the graph of \( g \) in the \( x \)-axis.
   (c) Write down the coordinates of the vertex of the graph of \( h \).

(Total 6 marks)
5. The quadratic equation $kx^2 + (k - 3)x + 1 = 0$ has two equal real roots.
   (a) Find the possible values of $k$. (5)
   (b) Write down the values of $k$ for which $x^2 + (k - 3)x + k = 0$ has two equal real roots. (2)
   (Total 7 marks)

6. Let $f(x) = 2x^2 + 4x - 6$.
   (a) Express $f(x)$ in the form $f(x) = 2(x - h)^2 + k$. (3)
   (b) Write down the equation of the axis of symmetry of the graph of $f$. (1)
   (c) Express $f(x)$ in the form $f(x) = 2(x - p)(x - q)$. (2)
   (Total 6 marks)
1. Let $f(x) = 3 \ln x$ and $g(x) = \ln 5x^3$.
   (a) Express $g(x)$ in the form $f(x) + \ln a$, where $a \in \mathbb{Z}^+$. (4)
   (b) The graph of $g$ is a transformation of the graph of $f$. Give a full geometric description of this transformation. (3)
   (Total 7 marks)

2. The functions $f(x)$ and $g(x)$ are defined by $f(x) = e^x$ and $g(x) = \ln (1+2x)$.
   (a) Write down $f^{-1}(x)$. (b) (i) Find $(f \circ g)(x)$. (ii) Find $(f \circ g)^{-1}(x)$. (Total 6 marks)
3. Let $f(x) = \log_a x, x > 0$.
   
   (a) Write down the value of $f(a), f(1), f(a^4)$. 

   (b) The diagram below shows part of the graph of $f$. On the same diagram, sketch the graph of $f^{-1}$. 

   (Total 6 marks)

4. Find the exact value of $x$ in each of the following equations.
   
   (a) $5^{x+1} = 625$
   
   (b) $\log_a (3x + 5) = 2$

   (Total 6 marks)
5. The diagram shows three graphs.

A is part of the graph of \( y = x \).

B is part of the graph of \( y = 2^x \).

C is the reflection of graph B in line A.

Write down

(a) the equation of C in the form \( y = f(x) \);

(b) the coordinates of the point where C cuts the x-axis.

(Total 4 marks)
1. The population of a city at the end of 1972 was 250 000. The population increases by 1.3% per year.
   (a) Write down the population at the end of 1973.
   (b) Find the population at the end of 2002.

2. Let $f(x) = \ln(x + 2), \ x > -2$ and $g(x) = e^{(x-4)}, \ x > 0$.
   (a) Write down the $x$-intercept of the graph of $f$.
   (b) (i) Write down $f(-1.999)$.
        (ii) Find the range of $f$.
   (c) Find the coordinates of the point of intersection of the graphs of $f$ and $g$. 

(Total 6 marks)
3. The area $A$ km$^2$ affected by a forest fire at time $t$ hours is given by $A = A_0 e^{kt}$.
   When $t = 5$, the area affected is 1 km$^2$ and the rate of change of the area is 0.2 km$^2$ h$^{-1}$.
   (a) Show that $k = 0.2$.  
      (4)
   (b) Given that $A_0 = \frac{1}{e}$, find the value of $t$ when 100 km$^2$ are affected.  
      (2)
      (Total 6 marks)

4. Solve the following equations.
   (a) $\ln (x + 2) = 3$.  
   (b) $10^{2x} = 500$.  
      (Total 6 marks)
5. A machine was purchased for $10000. Its value $V$ after $t$ years is given by $V = 100000e^{-0.3t}$. The machine must be replaced at the end of the year in which its value drops below $1500$. Determine in how many years the machine will need to be replaced.

(Total 6 marks)

6. The population $p$ of bacteria at time $t$ is given by $p = 100e^{0.05t}$. Calculate

(a) the value of $p$ when $t = 0$;

(b) the rate of increase of the population when $t = 10$.

(Total 6 marks)
7. The mass $m$ kg of a radio-active substance at time $t$ hours is given by $m = 4e^{-0.2t}$.

(a) Write down the initial mass.

(b) The mass is reduced to 1.5 kg. How long does this take?

(Total 6 marks)

8. $1000 is invested at 15% per annum interest, compounded monthly. Calculate the minimum number of months required for the value of the investment to exceed $3000.

(Total 6 marks)
9. Each year for the past five years the population of a certain country has increased at a steady rate of 2.7% per annum. The present population is 15.2 million.

(a) What was the population one year ago?
(b) What was the population five years ago?

(Total 4 marks)

10. A group of ten leopards is introduced into a game park. After $t$ years the number of leopards, $N$, is modelled by $N = 10 e^{0.4t}$.

(a) How many leopards are there after 2 years?
(b) How long will it take for the number of leopards to reach 100? Give your answers to an appropriate degree of accuracy.

Give your answers to an appropriate degree of accuracy.

(Total 4 marks)
11. A population of bacteria is growing at the rate of 2.3% per minute. How long will it take for the size of the population to double? Give your answer to the nearest minute.

(Total 4 marks)

12. The number of bacteria, \( n \), in a dish, after \( t \) minutes is given by \( n = 800e^{0.13t} \).

(a) Find the value of \( n \) when \( t = 0 \).

(b) Find the rate at which \( n \) is increasing when \( t = 15 \).

(c) After \( k \) minutes, the rate of increase in \( n \) is greater than 10 000 bacteria per minute. Find the least value of \( k \), where \( k \in \mathbb{Z} \).

(Total 8 marks)
13. Let \( f(x) = \log_3 \frac{x}{2} + \log_3 16 - \log_3 4 \), for \( x > 0 \).

(a) Show that \( f(x) = \log_3 2x \).

(b) Find the value of \( f(0.5) \) and of \( f(4.5) \).

The function \( f \) can also be written in the form \( f(x) = \frac{\ln ax}{\ln b} \).

(c) (i) Write down the value of \( a \) and of \( b \).

(ii) Hence on graph paper, sketch the graph of \( f \), for \(-5 \leq x \leq 5\), \(-5 \leq y \leq 5\), using a scale of 1 cm to 1 unit on each axis.

(iii) Write down the equation of the asymptote.

(d) Write down the value of \( f^{-1}(0) \).

The point A lies on the graph of \( f \). At A, \( x = 4.5 \).

(e) On your diagram, sketch the graph of \( f^{-1} \), noting clearly the image of point A.

(Total 16 marks)
A city is concerned about pollution, and decides to look at the number of people using taxis. At the end of the year 2000, there were 280 taxis in the city. After $n$ years the number of taxis, $T$, in the city is given by

\[ T = 280 \times 1.12^n. \]

(a)  
(i) Find the number of taxis in the city at the end of 2005.
(ii) Find the year in which the number of taxis is double the number of taxis there were at the end of 2000.

(b)  
At the end of 2000 there were 25 600 people in the city who used taxis.

After $n$ years the number of people, $P$, in the city who used taxis is given by

\[ P = \frac{2560000}{10 + 90e^{-0.1n}}. \]

(i) Find the value of $P$ at the end of 2005, giving your answer to the nearest whole number.
(ii) After seven complete years, will the value of $P$ be double its value at the end of 2000? Justify your answer.

(c)  
Let $R$ be the ratio of the number of people using taxis in the city to the number of taxis. The city will reduce the number of taxis if $R < 70$.

(i) Find the value of $R$ at the end of 2000.
(ii) After how many complete years will the city first reduce the number of taxis?

(Total 17 marks)
15. There were 1420 doctors working in a city on 1 January 1994. After $n$ years the number of doctors, $D$, working in the city is given by $D = 1420 + 100n$.

(a) (i) How many doctors were there working in the city at the start of 2004?
(ii) In what year were there first more than 2000 doctors working in the city?

At the beginning of 1994 the city had a population of 1.2 million. After $n$ years, the population, $P$, of the city is given by

$$P = 1200000 \times (1.025)^n.$$ 

(b) (i) Find the population $P$ at the beginning of 2004.
(ii) Calculate the percentage growth in population between 1 January 1994 and 1 January 2004.
(iii) In what year will the population first become greater than 2 million?

(c) (i) What was the average number of people per doctor at the beginning of 1994?
(ii) After how many complete years will the number of people per doctor first fall below 600?

(Total 15 marks)
16. Michele invested 1500 francs at an annual rate of interest of 5.25 percent, compounded annually.

(a) Find the value of Michele’s investment after 3 years. Give your answer to the nearest franc. (3)

(b) How many complete years will it take for Michele’s initial investment to double in value? (3)

(c) What should the interest rate be if Michele’s initial investment were to double in value in 10 years? (4)

(Total 10 marks)
17. Initially a tank contains 10 000 litres of liquid. At the time $t = 0$ minutes a tap is opened, and liquid then flows out of the tank. The volume of liquid, $V$ litres, which remains in the tank after $t$ minutes is given by

$$V = 10 000 (0.933^t).$$

(a) Find the value of $V$ after 5 minutes.

(b) Find how long, to the nearest second, it takes for half of the initial amount of liquid to flow out of the tank.

(c) The tank is regarded as effectively empty when 95% of the liquid has flowed out. Show that it takes almost three-quarters of an hour for this to happen.

(d) Find the value of $10 000 - V$ when $t = 0.001$ minutes.

(Total 7 marks)
1. Solve \( \log_2 x + \log_2(x - 2) = 3 \), for \( x > 2 \).

(Total 7 marks)

2. Let \( f(x) = \sqrt{3} e^{2x} \sin x + e^{2x} \cos x \), for \( 0 \leq x \leq \pi \). Given that \( \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \), solve the equation \( f(x) = 0 \).

(Total 6 marks)
3. Solve the following equations.

(a) \( \log_{10} 49 = 2 \)  
   \[ \text{Total 13 marks} \]  

(b) \( \log_2 8 = x \)  
   \[ \text{Total 13 marks} \]  

(c) \( \log_{25} x = -\frac{1}{2} \)  
   \[ \text{Total 13 marks} \]  

(d) \( \log_2 x + \log_2(x - 7) = 3 \)  
   \[ \text{Total 13 marks} \]  

4. Solve the equation \( \log_9 81 + \log_9 \frac{1}{9} + \log_9 3 = \log_9 x. \)  
   \[ \text{Total 4 marks} \]
2.6 Solving Equations

1. (a) Given that \((2^x)^2 + (2^x) - 12\) can be written as \((2^x + a)(2^x + b)\), where \(a, b \in \mathbb{Z}\), find the value of \(a\) and of \(b\).

(b) Hence find the **exact** solution of the equation \((2^x)^2 + (2^x) - 12 = 0\), and explain why there is only one solution.

(Total 6 marks)

2. Solve the equation \(\log_{27} x = 1 - \log_{27} (x - 0.4)\).

(Total 6 marks)
3. Solve the equation \( e^x = 5 - 2x \), giving your answer correct to \textbf{four} significant figures.

(Total 6 marks)

4. (a) Sketch, on the given axes, the graphs of \( y = x^2 \) and \( y = \sin x \) for \(-1 \leq x \leq 2\).

(b) Find the positive solution of the equation \( x^2 = \sin x \), giving your answer correct to 6 significant figures.

(Total 4 marks)
5. Let \( g(x) = \frac{1}{2} x \sin x \), for \( 0 \leq x \leq 4 \).
   
   (a) Sketch the graph of \( g \) on the following set of axes. 
   
   (b) Hence find the value of \( x \) for which \( g(x) = -1 \). 
   
   (Total 6 marks)

6. Solve the equation \( e^x = 4 \sin x \), for \( 0 \leq x \leq 2\pi \). 
   
   (Total 5 marks)

7. Let \( f(x) = 4 \tan^2 x - 4 \sin x \), for \( -\frac{\pi}{3} \leq x \leq \frac{\pi}{3} \).
   
   (a) On the grid, sketch the graph of \( y = f(x) \). 
   
   (b) Solve the equation \( f(x) = 1 \). 
   
   (Total 6 marks)
8. The diagram below shows a quadrilateral ABCD with obtuse angles $\hat{ABC}$ and $\hat{ADC}$.

$AB = 5\, \text{cm}, \ BC = 4\, \text{cm}, \ CD = 4\, \text{cm}, \ AD = 4\, \text{cm}, \ \hat{BAC} = 30^\circ, \ \hat{ABC} = x^\circ, \ \hat{ADC} = y^\circ$.

(a) Use the cosine rule to show that $AC = \sqrt{41 - 40 \cos x}$.

(b) Use the sine rule in triangle ABC to find another expression for AC.

(c) (i) Hence, find $x$, giving your answer to two decimal places.

(ii) Find AC.

(d) (i) Find $y$.

(ii) Hence, or otherwise, find the area of triangle ACD.

(Total 14 marks)
9. A farmer owns a triangular field ABC. One side of the triangle, [AC], is 104 m, a second side, [AB], is 65 m and the angle between these two sides is 60°.

(a) Use the cosine rule to calculate the length of the third side of the field. (3)

(b) Given that \( \sin 60° = \frac{\sqrt{3}}{2} \), find the area of the field in the form \( 3p\sqrt{3} \) where \( p \) is an integer. (3)

Let D be a point on [BC] such that [AD] bisects the 60° angle. The farmer divides the field into two parts \( A_1 \) and \( A_2 \) by constructing a straight fence [AD] of length \( x \) metres, as shown on the diagram below.

(c) (i) Show that the area of \( A_1 \) is given by \( \frac{65x}{4} \).

(ii) Find a similar expression for the area of \( A_2 \).

(iii) **Hence**, find the value of \( x \) in the form \( q\sqrt{3} \), where \( q \) is an integer. (7)

(d) (i) Explain why \( \sin \hat{A} DC = \sin \hat{A} DB \).

(ii) Use the result of part (i) and the sine rule to show that \( \frac{BD}{DC} = \frac{5}{8} \). (5)

(Total 18 marks)