Functions and Equations
workbook markscheme

~ categorized past IB Paper 1 and Paper 2 examination questions ~

IB DP Mathematics Standard Level

Topic 2
This workbook contains past Paper 1 and Paper 2 IB examination questions categorized according to major concepts in this topic.

Contents

- **2.1 Composite and Inverse Functions**
  - Paper 1 Questions
    - 24 questions; 142 marks
  - Paper 2 Questions
    - 4 questions; 24 marks

- **2.2 Graphs of Functions**
  - Paper 1 Questions
    - 2 questions; 12 marks
  - Paper 2 Questions
    - 9 questions; 50 marks

- **2.3 Transformation of Graphs**
  - Paper 1 Questions
    - 15 questions 105 marks
  - Paper 2 Questions
    - 3 questions; 26 marks

- **2.4 Quadratic Functions**
  - Paper 1 Questions
    - 24 questions 130 marks
  - Paper 2 Questions
    - 6 questions; 55 marks

- **2.5 Logarithmic and Exponential Functions**
  - Paper 1 Questions
    - 5 questions 29 marks
  - Paper 2 Questions
    - 17 questions; 136 marks

- **2.6 Solving Equations**
  - Paper 1 Questions
    - 4 questions 30 marks
  - Paper 2 Questions
    - 9 questions; 71 marks

The use of GDC is not permitted for Paper 1 but is required for Paper 2 questions.
2.1 Composite and Inverse Functions

1. (a) attempt to form composite
   
   e.g. \( g(7 - 2x), 7 - 2x + 3 \)
   
   \( (g \circ f)(x) = 10 - 2x \)
   
   A1 N2 2
   
   (b) \( g^{-1}(x) = x - 3 \)
   
   A1 N1 1
   
   (c) **METHOD 1**
   
   valid approach
   
   e.g. \( g^{-1}(5), 2, f(5) \)
   
   \( f(2) = 3 \)
   
   A1 N2 2
   
   **METHOD 2**
   
   attempt to form composite of \( f \) and \( g^{-1} \)
   
   e.g. \( (f \circ g^{-1})(x) = 7 - 2(x - 3), 13 - 2x \)
   
   \( (f \circ g^{-1})(5) = 3 \)
   
   A1 N2 2

2. (a) interchanging \( x \) and \( y \) (seen anywhere)
   
   e.g. \( x = \log \sqrt{y} \) (accept any base)
   
   evidence of correct manipulation
   
   e.g. \( 3^x = \sqrt[3]{y}, 3^y = x^2, x = \frac{1}{2} \log_3 y, 2y = \log_3 x \)
   
   \( f^{-1}(x) = 3^{2x} \)
   
   AG N0
   
   (b) \( y > 0, f^{-1}(x) > 0 \)
   
   A1 N1
   
   (c) **METHOD 1**
   
   finding \( g(2) = \log_3 2 \) (seen anywhere)
   
   attempt to substitute
   
   e.g. \( (f^{-1} \circ g)(2) = 3^{\log_3 2} \)
   
   evidence of using log or index rule
   
   e.g. \( (f^{-1} \circ g)(2) = 3^{\log_3 4}, 3^{\log_3 2^2} \)
   
   \( (f^{-1} \circ g)(2) = 4 \)
   
   A1 N1
   
   **METHOD 2**
   
   attempt to form composite (in any order)
   
   e.g. \( (f^{-1} \circ g)(x) = 3^{2^{\log_3 x}} \)
   
   evidence of using log or index rule
   
   e.g. \( (f^{-1} \circ g)(x) = 3^{\log_3 x^2}, 3^{\log_3 x^2} \)
   
   \( (f^{-1} \circ g)(x) = x^2 \)
   
   A1
   
   \( (f^{-1} \circ g)(2) = 4 \)
   
   A1 N1

[5]

[7]
3. (a) \( f \left( \frac{\pi}{2} \right) = \cos \pi = -1 \) 
(b) \( (g \circ f) \left( \frac{\pi}{2} \right) = g(-1) = 2(-1)^2 - 1 = 1 \) 
(c) \( (g \circ f)(x) = 2(\cos (2x))^2 - 1 = 2 \cos^2(2x) - 1 \) 
\( (g \circ f)(x) = 4x \)
\( k = 4 \)

4. (a) for interchanging \( x \) and \( y \) (may be done later) 
\( e.g. \ x = 2y - 3 \) 
\( g^{-1}(x) = \frac{x+3}{2} \) 
\( (f \circ g)(4) = (2 \times 4 - 3)^2 = 25 \) 

5. (a) (i) \( g(0) = e^0 - 2 = -1 \) 
(ii) METHOD 1 
substituting answer from (i) 
\( e.g. \ (f \circ g)(0) = f(-1) \) 
\( f(-1) = 2(-1)^3 + 3 \) 
\( f(-1) = 1 \) 
METHOD 2 
attempt to find \( (f \circ g)(x) \) 
\( e.g. \ (f \circ g)(x) = f(e^{3x} - 2) = 2(e^{3x} - 2)^3 + 3 \) 
\( (f \circ g)(0) = 1 \) 

(b) interchanging \( x \) and \( y \) (seen anywhere) 
\( e.g. \ x = 2y^3 + 3 \) 
\( f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}} \)
6. (a) **METHOD 1**

\[ \ln (x + 5) + \ln 2 = \ln (2(x + 5)) (= \ln (2x + 10)) \]  
(A1)  
interchanging \( x \) and \( y \) (seen anywhere)  
(M1)  
e.g. \( x = \ln (2y + 10) \)  
evidence of correct manipulation  
(A1)  
e.g. \( e^x = 2y + 10 \)  
\[ f^{-1}(x) = \frac{e^x - 10}{2} \]  
A1 N2

**METHOD 2**

\[ y = \ln (x + 5) + \ln 2 \]  
\[ y - \ln 2 = \ln (x + 5) \]  
(A1)  
evidence of correct manipulation  
(A1)  
e.g. \( e^{y - \ln 2} = x + 5 \)  
interchanging \( x \) and \( y \) (seen anywhere)  
(M1)  
e.g. \( e^{x - \ln 2} = y + 5 \)  
\[ f^{-1}(x) = e^{x - \ln 2} - 5 \]  
A1 N2

(b) evidence of composition in correct order  
(M1)  
e.g. \( (g \circ f)(x) = g(\ln (x + 5) + \ln 2) \)  
\[ = e^{\ln (2(x + 5))} = 2(x + 5) \]  
\[ (g \circ f)(x) = 2x + 10 \]  
A1A1 N2

7. (a) (i) \( \sqrt{6} \)  
A1 N1  
(ii) 9  
A1 N1  
(iii) 0  
A1 N1

(b) \( x < 5 \)  
A2 N2

(c) \( (g \circ f)(x) = (\sqrt{x - 5})^2 \)  
(M1)  
\[ = x - 5 \]  
A1 N2

8. (a) **METHOD 1**

\[ f(3) = \sqrt[7]{7} \]  
(A1)  
\[ (g \circ f)(3) = 7 \]  
A1 N2

**METHOD 2**

\[ (g \circ f)(x) = \sqrt{x + 4}^2 = x + 4 \]  
(A1)  
\[ (g \circ f)(3) = 7 \]  
A1 N2

(b) For interchanging \( x \) and \( y \) (seen anywhere)  
Evidence of correct manipulation  
A1  
e.g. \( x = \sqrt{y + 4}, x^2 = y + 4 \)  
\[ f^{-1}(x) = x^2 - 4 \]  
A1 N2

(c) \( x \geq 0 \)  
A1 N1
9. (a) **METHOD 1**

For \( f(-2) = -12 \) (A1)

\((g * f)(-2) = g(-12) = -24\)

**METHOD 2**

\((g * f)(x) = 2x^3 - 8\) (A1)

\((g * f)(-2) = -24\)

(b) Interchanging \( x \) and \( y \) (may be done later) (M1)

\( x = y^3 - 4 \)

\( f^{-1}(x) = \sqrt[3]{x+4} \)

[6]

10. (a) Evidence of attempting to form composition (M1)

Correct substitution \((h * g)(x) = \frac{5(3x-2)}{(3x-2) - 4}\)

= \( \frac{5(3x-2)}{3x-6} \left( = \frac{15x-10}{3x-6} \right) \left( = \frac{5(3x-2)}{3(x-2)} \right) \)

A1 N2

(b) Evidence of using numerator = 0 (M1)

\( eg \frac{15x-10}{3x-6} = 0 \) (\( 3x - 2 = 0 \))

\( x = \frac{2}{3} \) (=0.667)

A2 N3

11. (a) \((f \circ g): x \mapsto 3(x + 2) = 3x + 6\)

(b) **METHOD 1**

\( f^{-1}(x) = \frac{x}{3} \quad g^{-1}(x) = x - 2 \) (M1)

\( f^{-1}(18) = \frac{18}{3} \)

\( g^{-1}(18) = 18 - 2 \)

\( f^{-1}(18) + g^{-1}(18) = 6 + 16 \) A1

\( f^{-1}(18) + g^{-1}(18) = 22 \) AG 4

**METHOD 2**

\( 3x = 18, x + 2 = 18 \) (M1)

\( x = 6, x = 16 \) A1A1

\( f^{-1}(18) + g^{-1}(18) = 6 + 16 \)

\( f^{-1}(18) + g^{-1}(18) = 22 \)

AG 4

[6]
12. (a) METHOD 1
\[ (f \circ g) (4) = f(g(4)) = f(1) \]  
\[ = 2 \]  
(M1)  
(A1) (C2)

METHOD 2
\[ (f \circ g) (x) = \frac{2}{x - 3} \]  
(M1)  
\[ (f \circ g) (4) = 2 \]  
(A1) (C2)

(b) Let \[ y = \frac{1}{x - 3} \]

Correct simplification \[ y(x - 3) = 1 \]  
\[ \left( x - 3 = \frac{1}{y} \right) \]  
(A1)  
\[ x = \frac{1}{y} + 3 \]  
(A1)  
\[ \left( \frac{1 + 3y}{y} \right) \]  
(C3)

Interchanging \( x \) and \( y \) (may happen earlier)  
(M1)

(c) \( x \neq 0 \) (\( \mathbb{R} \setminus \{0\} \) etc)  
(A1) (C1)

13. (a) \[ y = 2x + 1 \]
\[ x = 2y + 1 \]  
(M1)
\[ \frac{x - 1}{2} = y \]

\[ f^{-1}(x) = \frac{x - 1}{2} \]  
(A1) (C2)

(b) \[ g(f(-2)) = g(-3) \]  
\[ = 3(-3)^2 - 4 \]  
\[ = 23 \]  
(A1) (C2)

(c) \[ f(g(x)) = f(3x^2 - 4) \]  
\[ = 2(3x^2 - 4) + 1 \]  
\[ = 6x^2 - 7 \]  
(A1) (A1) (C2)

14. (a) \[ x = e^{-y} \]  
\[ \ln x = -y \]  
\[ y = f^{-1}(x) = -\ln x \]  
(A1) (A1) (C3)

(b) \[ (g \cdot f)(x) = g(e^{-x}) \]  
(M1)
\[ = \frac{e^{-x}}{1 + e^{-x}} \]  
(A2) (C3)
15. (a) \( f(3) = 2^3 \) 
\[
(g \circ f)(3) = \frac{2^3}{2^3 - 2} = \frac{8}{6} = \frac{4}{3}
\]
\( (g \circ f)(3) = \frac{4}{3} \) (C3)

(b) \[ x = \frac{y}{y - 2} \quad \text{(M1)} \]
\[ x (y - 2) = y \Rightarrow y (x - 1) = 2x \]
\[ \Rightarrow y = \frac{2x}{x - 1} \quad \text{(A1)} \]
\[ y = \frac{10}{5 - 1} = 2.5 \quad \text{(A1) (C3)} \]

Note: Interchanging \( x \) and \( y \) may take place at any time.

16. (a) \[ y = \frac{6 - x}{2} \]
\[ \Rightarrow x = \frac{6 - y}{2} \quad \text{(M1)} \]
\[ \Rightarrow y = 6 - 2x = g^{-1}(x) \quad \text{(A1) (C2)} \]

(b) \[ (f \circ g^{-1})(x) = 4[(6 - 2x) - 1] = 4(5 - 2x) = 20 - 8x \]
\[ 20 - 8x = 4 \Rightarrow 8x = 16 \Rightarrow x = 2 \quad \text{(M1)(A1) (A1) (C4)} \]

17. (a) \[ f^{-1}(2) \Rightarrow 3x + 5 = 2 \]
\[ x = -1 \quad \text{(M1) (A1) (C2)} \]

(b) \[ g(f(-4)) = g(-12 + 5) = g(-7) = 2(1 + 7) = 16 \quad \text{(A1) (A1) (C2)} \]

18. \[ (g \circ f)(x) = 0 \Rightarrow 2 \cos x + 1 = 0 \]
\[ \Rightarrow \cos x = -\frac{1}{2} \quad \text{(M1) (A1)} \]
\[ x = \frac{2\pi}{3}, \frac{4\pi}{3} \quad \text{(A1)(A1) (C4)} \]

Note: Accept 120°, 240°.

19. \[ x = g^{-1}(f(0.25)) \]
\[ = \log_2((0.25)^{1/2}) \]
\[ = \log_2\left(\frac{1}{2}\right) \]
\[ = -1 \quad \text{(A1) (A1)} \]

[4]
20. (a) (i) interchanging $x$ and $y$ (seen anywhere) correct manipulation 
\[ f^{-1}(x) = \ln x - 3 \] 
(ii) $x > 0$ 
(b) collecting like terms; using laws of logs 
\[ \frac{3}{2} = e^\frac{3}{2} \] (A1)

21. (a) **METHOD 1**
recognizing that $f(8) = 1$ (M1)
e.g. $1 = k \log_2 8$
recognizing that $\log_2 8 = 3$ (A1)
e.g. $1 = 3k$
\[ k = \frac{1}{3} \] (A1)

**METHOD 2**
attempt to find the inverse of $f(x) = k \log_2 x$ (M1)
e.g. $x = k \log_2 y, y = 2^x$
substituting 1 and 8 (M1)
e.g. $1 = k \log_2 8, \frac{1}{2^x} = 8$
\[ k = \frac{1}{\log_2 8} \left( k = \frac{1}{3} \right) \] (A1)

(b) **METHOD 1**
recognizing that $f(x) = \frac{2}{3}$ (M1)
e.g. $\frac{2}{3} = \frac{1}{3} \log_2 x$
\[ \log_2 x = 2 \] (A1)
\[ f^{-1}\left(\frac{2}{3}\right) = 4 \text{ (accept } x = 4) \] (A2)

**METHOD 2**
attempt to find inverse of $f(x) = \frac{1}{3} \log_2 x$ (M1)
e.g. interchanging $x$ and $y$, substituting $k = \frac{1}{3}$ into $y = 2^x$
correct inverse (A1)
e.g. $f^{-1}(x) = 2^{3x}, 2^{3x}$
\[ f^{-1}\left(\frac{2}{3}\right) = 4 \] (A2)
22. (a) interchanging $x$ and $y$ (may happen later) $x = e^{y-11} - 8$  
\[ e^{y-11} = x + 8 \]  
\[ \ln(e^{y-11}) = \ln(x + 8) \]  
\[ f^{-1}(x) = \ln(x + 8) + 11 \]  
(b) Domain is $x > -8$

23. (a) $a = 3, b = 4$  
\[ f(x) = (x - 3)^2 + 4 \]  
(b) $y = (x - 3)^2 + 4$  
\[ \text{METHOD 1} \]  
\[ x = (y - 3)^2 + 4 \]  
\[ x - 4 = (y - 3)^2 \]  
\[ \sqrt{x - 4} = y - 3 \]  
\[ y = \sqrt{x - 4} + 3 \]  
\[ \text{METHOD 2} \]  
\[ y - 4 = (x - 3)^2 \]  
\[ \sqrt{y - 4} = x - 3 \]  
\[ \sqrt{y - 4} + 3 = x \]  
\[ y = \sqrt{y - 4} + 3 \]  
\[ \Rightarrow f^{-1}(x) = \sqrt{x - 4} + 3 \]  
(c) $x \geq 4$

24. $f(x) = 2e^{3x}$. Let $x = 2e^{3y}$  
\[ \Rightarrow \frac{x}{2} = e^{3y} \]  
\[ \Rightarrow \ln\left(\frac{x}{2}\right) = 3y \]  
\[ \Rightarrow y = \frac{1}{3} \ln\left(\frac{x}{2}\right) \]  
\[ \Rightarrow f^{-1}(x) = \frac{1}{3} \ln\left(\frac{x}{2}\right) \]  
\[ \text{that is } f^{-1}(x) = \frac{1}{3} \ln\left(\frac{x}{2}\right) \]
1. (a) 

(b) \[ 0 \leq y \leq 3.5 \]

(c) interchanging \( x \) and \( y \) (seen anywhere) 
   
   \[ e.g. \ x = e^{0.5y} \]
   
   evidence of changing to log form 
   
   \[ e.g. \ \ln x = 0.5y, \ \ln x = \ln e^{0.5y} \ (any \ base), \ \ln x = 0.5 \ y \ln e \ (any \ base) \]
   
   \[ f^{-1}(x) = 2 \ln x \]

\[ [7] \]

2. (a) attempt to form composite 
   
   \[ e.g. \ f(2x - 5) \]
   
   \[ h(x) = 6x - 15 \]

(b) interchanging \( x \) and \( y \) 
   
   evidence of correct manipulation 
   
   \[ e.g. \ y + 15 - 6x, \ \frac{x}{6} = y - \frac{5}{2} \]
   
   \[ h^{-1}(x) = \frac{x + 15}{6} \]

\[ [5] \]

3. (a) attempt to form any composition (even if order is reversed) 
   
   correct composition \( h(x) = g\left(\frac{3x}{2} + 1\right) \)
   
   \[ h(x) = 4 \cos\left(\frac{3x}{2} + 1\right) - 1 \left( 4 \cos\left(\frac{1}{2} x + \frac{1}{3}\right) - 1, 4 \cos\left(\frac{3x + 2}{6}\right) - 1 \right) \]

(b) period is \( 4\pi(12.6) \)

(c) range is \( -5 \leq h(x) \leq 3 \ ([-5, 3]) \)

\[ [6] \]
4. (a) \((f \circ g): x \mapsto 3(x + 2) \quad (= 3x + 6)\)

(b) **METHOD 1**

Evidence of finding inverse functions

\[ e.g.\ f^{-1}(x) = \frac{x}{3} \quad g^{-1}(x) = x - 2 \]

\[ f^{-1}(18) = \frac{18}{3} = 6 \quad \text{(A1)} \]

\[ g^{-1}(18) = 18 - 2 = 16 \quad \text{(A1)} \]

\[ f^{-1}(18) + g^{-1}(18) = 6 + 16 = 22 \quad \text{A1 N3} \]

**METHOD 2**

Evidence of solving equations

\[ e.g.\ 3x = 18, \ x + 2 = 18 \]

\[ x = 6, \ x = 16 \quad \text{(A1)(A1)} \]

\[ f^{-1}(18) + g^{-1}(18) = 6 + 16 = 22 \quad \text{A1 N3} \]
2.2 Graphs of Functions

1. (a) \( y = -2x + 3 \)

gradient of line \( L = -2 \)  

(b) METHOD 1

\( (y - y_1) = m(x - x_1) \Rightarrow (y - (-4)) = -2(x - 6) \)

\( y + 4 = -2x + 12 \)

\( y = -2x + 8 \)  

METHOD 2

Substituting the point \((6, -4)\) in \( y = mx + c \), ie \( -4 = -2(6) + b \)

\( b = 8 \)

\( y = -2x + 8 \)  

(c) when line \( L \) cuts the x-axis, \( y = 0 \)

\( y = -2x + 8 \)

\( x = 4 \)

2.

<table>
<thead>
<tr>
<th>sketch</th>
<th>relation letters</th>
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<tbody>
<tr>
<td>(i)</td>
<td>A</td>
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<td>(ii)</td>
<td>C</td>
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<td>B</td>
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(A1)(A1) (C2)

(A1)(A1) (C2)

(A1)(A1) (C2)

[6]
2.2 Graphs of Functions

1. (a) [Graph of a function]

(b) $3.19$

(c) $p = 1.89, q = 3.19$

2. (a) [Graph of a function]

(b) $x = 3, x = -3$

(c) $y \geq 1$
3. 

Note: Award no marks if candidates work in degrees.

(a) \[ (A1)(A1)(A1)(A1) \] (C4)

(b) \[ 1.26, 2.26 \] (A1)(A1) (C1)(C1)

4. 

(a) \[ (A1)(A1)(C1)(C1) \]

(b) \[ x = -1.29 \] (A2) (C2)

5. 

(a) \[ y = 0 \Rightarrow x = 0 \text{ or } \sin \frac{x}{3} = 0 \] (M1)

\[ \Rightarrow \frac{x}{3} = 0, \pi \]

\[ \Rightarrow x = 0, 3\pi \]

\[ m = 10 \] (A1)

(b) \[ y_{\text{max}} = 5.46 \text{ (or between 5 and 6)} \] (M1)

\[ \Rightarrow n = 6 \] (A1) (C2)
6. From sketch of graph \( y = 4 \sin \left( 3x + \frac{\pi}{2} \right) \) (M2)
or by observing \(|\sin \theta| \leq 1\).
\[ k > 4, \ k < -4 \] (A1)(A1)(C2)(C2)

7. (a) Attempt to substitute points into the function (M1)
e.g. \(-8 = p(-2)^3 + q(-2)^2 + r(-2)\), one correct equation
\[-8 = -8p + 4q - 2r, \ -2 = p + q + r, \ 0 = 8p + 4q + 2r\] (A1A1A1 N3)

(b) \( x = -1.32, x = 1.68 \) (accept \( x = -1.41, x = 1.39 \) if working in degrees) A1A1 N2

(c) \(-1.32 < x < 1.68 \) (accept \(-1.41 < x < 1.39 \) if working in degrees) A2 N2

8. (a) Attempt to substitute points into the function (M1)
e.g. \(-8 = p(-2)^3 + q(-2)^2 + r(-2)\), one correct equation
\[-8 = -8p + 4q - 2r, \ -2 = p + q + r, \ 0 = 8p + 4q + 2r\] A1A1A1 N4

(b) Attempt to solve system (M1)
e.g. inverse of a matrix, substitution
\( p = 1, q = -1, r = -2 \) A2 N3
9. (a)

(b) $x = 3, x = -3$

(c) $y \geq 1$
2.3 Transformation of Graphs

1. (a) attempt to form composition (in any order) \((f \circ g)(x) = (x - 1)^2 + 4 \quad (x^2 - 2x + 5)\) (M1) A1 N2

(b) **METHOD 1**

vertex of \(f \circ g\) at (1, 4) (A1)

evidence of appropriate approach (M1)

e.g. adding \(\begin{pmatrix} 3 \\ -1 \end{pmatrix}\) to the coordinates of the vertex of \(f \circ g\)

vertex of \(h\) at (4, 3) A1 N3

**METHOD 2**

attempt to find \(h(x)\) (M1)

e.g. \(((x - 3) - 1)^2 + 4 - 1, h(x) = (f \circ g)(x - 3) - 1\)

\(h(x) = (x - 4)^2 + 3\) (A1)

vertex of \(h\) at (4, 3) A1 N3

(c) evidence of appropriate approach (M1)

e.g. \((x - 4)^2 + 3, (x - 3)^2 - 2(x - 3) + 5 - 1\)

simplifying A1

e.g. \(h(x) = x^2 - 8x + 16 + 3, x^2 - 6x + 9 - 2x + 6 + 4\)

\(h(x) = x^2 - 8x + 19\) AG N0

(d) **METHOD 1**

equating functions to find intersection point (M1)

e.g. \(x^2 - 8x + 19 = 2x - 6, y = h(x)\)

\(x^2 - 10x + 25 = 0\) A1

evidence of appropriate approach to solve (M1)

e.g. factorizing, quadratic formula

appropriate working A1

e.g. \((x - 5)^2 = 0\)

\(x = 5 (p = 5)\) A1 N3

**METHOD 2**

attempt to find \(h'(x)\) (M1)

\(h'(x) = 2x - 8\) A1

recognizing that the gradient of the tangent is the derivative (M1)

e.g. gradient at \(p = 2\)

\(2x - 8 = 2 (2x = 10)\) A1

\(x = 5\) A1 N3

[12]
2. (a)

(b) evidence of appropriate approach

\[ e.g. \text{ reference to any horizontal shift and/or stretch factor, } x = 3 + 1, y = \frac{1}{2} \times 2 \]

P is (4, 1) (accept \( x = 4, y = 1 \))

\[ \text{A1A1 \ N3} \]

[5]

3. (a) in any order

translated 1 unit to the right \[ \text{A1 \ N1} \]
stretched vertically by factor 2 \[ \text{A1 \ N1} \]

(b) METHOD 1

Finding coordinates of image on \( g \)

\[ e.g. -1 + 1 = 0, 1 \times 2 = 2, (-1, 1) \rightarrow (-1 + 1, 2 \times 1), (0, 2) \]

P is (3, 0)

\[ \text{A1A1 \ N4} \]

METHOD 2

\[ h(x) = 2(x - 4)^2 - 2 \]

P is (3, 0)

\[ \text{A1A1 \ N4} \]

[6]

4. (a)

(b) (i) \( g(-3) = f(0) \)

\[ f(0) = -1.5 \]

(ii) translation (accept shift, slide, etc.) of \( \begin{pmatrix} -3 \\ 0 \end{pmatrix} \)

\[ \text{A1A1 \ N2} \]

[6]
5. (a) (i) attempt to substitute  
\[ e.g. \ a = \frac{29 - 15}{2} \]  
\[ a = 7 \text{ (accept } a = -7) \]  
A1 N2

(ii) period = 12  
\[ b = \frac{2\pi}{12} \]  
A1

\[ b = \frac{\pi}{6} \]  
AG N0

(iii) attempt to substitute  
\[ e.g. \ d = \frac{29 + 15}{2} \]  
\[ d = 22 \]  
A1 N2

(iv) \ c = 3 \text{ (accept } c = 9 \text{ from } a = -7) \]  
A1 N1

(b) stretch takes 3 to 1.5  
translation maps (1.5, 29) to (4.5, 19) (so \( M' \) is (4.5, 19))  
A1 N2

(c) \[ g(t) = 7 \cos \left( \frac{\pi}{3} (t - 4.5) + 12 \right) \]  
A1A2A1 N4

(d) translation \[ \begin{pmatrix} -3 \\ 10 \end{pmatrix} \]  
horizontal stretch of a scale factor of 2  
completely correct description, in correct order  
e.g. translation \[ \begin{pmatrix} -3 \\ 10 \end{pmatrix} \] then horizontal stretch of a scale factor of 2

6. (a)
(b) \[ g(x) = \frac{1}{x-2} + 3 \]

(c) (i) Evidence of using \( x = 0 \) \( \left(g(0) = -\frac{1}{2} + 3\right) \)

\[ y = \frac{5}{2} \quad (= 2.5) \]

Evidence of solving \( y = 0 \) \( (1 + 3(x - 2) = 0) \)

\[ 1 + 3x - 6 = 0 \]

\[ 3x = 5 \]

\[ x = \frac{5}{3} \]

Intercepts are \( x = \frac{5}{3} \), \( y = \frac{5}{2} \) (accept \( \left(\frac{5}{3}, 0\right) \), \( \left(0, \frac{5}{2}\right) \))

(ii) \( x = 2 \)

\( y = 3 \)

(iii)
7. (a) 

(b) \( y = 1 \) (must be an equation) 
(c) \((0, 3)\) 

8. (a) D 
(b) C 
(c) A 

9. (a) (i) 

(ii) 

(b) \( A'(3, 2) \) (Accept \( x = 3, \ y = 2 \))
10. (a) \( g(x) = 2f(x-1) \)

<table>
<thead>
<tr>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
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<tr>
<td>( x-1 )</td>
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<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( f(x-1) )</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\[

g(0) = 2f(-1) = 6 \quad \text{(A1) (C1)} \\
g(1) = 2f(0) = 4 \quad \text{(A1) (C1)} \\
g(2) = 2f(1) = 0 \quad \text{(A1) (C1)} \\
g(3) = 2f(2) = 2 \quad \text{(A1) (C1)} \\
\]

(b) Graph passing through (0, 6), (1, 4), (2, 0), (3, 2) \( \text{Correct shape.} \) \( \text{(C2)} \)
11. 
(a) 
(b) Minimum: \( \left(1, \frac{1}{2}\right) \) 
Maximum: \((2, 2)\) 

12. 
(a) Correct vertical shift 
Coordinates of the images (see diagram) 
(b) Asymptote: \( y = -3 \) 

13. 
(a) \( y = (x - 1)^2 \) 
(b) \( y = 4(x - 1)^2 \) 
(c) \( y = 4(x - 1)^2 + 3 \) 

14. 
(a) I 
(b) III 
(c) IV
15. (a) 

(b) | Description of transformation | Diagram letter |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
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<tbody>
<tr>
<td>Horizontal stretch with scale factor 1.5</td>
<td>C</td>
</tr>
<tr>
<td>Maps $f$ to $f(x) + 1$</td>
<td>D</td>
</tr>
</tbody>
</table>

(c) translation (accept move/shift/slide etc.) with vector A1A1 N2

[6]
## 2.3 Transformation of Graphs

### 1. (a) $f(x) = x^2 - 2x - 3$

- Evidence of solving $f'(x) = 0$
  - e.g. $x^2 - 2x - 3 = 0$
- Evidence of correct working
  - e.g. $(x + 1)(x - 3) = 0$
  - $x = -1$ (ignore $x = 3$)

- Evidence of substituting their negative $x$-value into $f(x)$
  - e.g. $\frac{2 \pm \sqrt{16}}{2}$

- $y = \frac{5}{3}$

- Coordinates are $\left(-1, \frac{5}{3}\right)$

(b) (i) $(-3, -9)$

(ii) $(1, -4)$

(iii) Reflection gives $(3, 9)$

- Stretch gives $\left(\frac{3}{2}, 9\right)$

### 2. (a) (i) $p = 2$

(ii) $10 = \frac{q}{3 - 2}$ (or equivalent)

- $q = 10$

(b) Reflection, in $x$-axis
3. 

(a) \((A1)(A1) (C2)\)

(b) \((A1)(A3) (C4)\)

(a) Note: Award \((A1)\) for the correct line, \((A1)\) for using the given domain.

(b) Correct domain \((A1)\)

EITHER

The correct line drawn \((A3)\)

OR

\[
g(x) = f(x + 3) - 2 = (2(x + 3) + 1) - 2 = 2x + 5 \quad (M1)
\]

Candidate’s line drawn \((A1)\)

OR

\[
g(-3) = -1 \quad g(-1) = 3 \quad (A1)(A1)
\]

Line joining \(g(-3)\) and \(g(-1)\) drawn \((A1)\)
1. (a) valid approach  
   \[ b^2 - 4ac, \Delta = 0, (-4k)^2 - 4(2k)(1) \]  
   correct equation  
   \[ (-4k)^2 - 4(2k)(1) = 0, 16k^2 = 8k, 2k^2 - k = 0 \]  
   correct manipulation  
   \[ 8k(2k-1), \frac{8 \pm \sqrt{64}}{32} \]  
   \[ k = \frac{1}{2} \]  
   (b) recognizing vertex is on the x-axis  
   \[ (1, 0), \text{sketch of parabola opening upward from the x-axis} \]  
   \[ P \geq 0 \]  

2. (a) evidence of setting function to zero  
   \[ f(x) = 0, 8x = 2x^2 \]  
   evidence of correct working  
   \[ 0 = 2x(4-x), \frac{-8 \pm \sqrt{64}}{-4} \]  
   x-intercepts are at 4 and 0 (accept (4, 0) and (0, 0), or \( x = 4, x = 0 \))  
   (b) (i) \( x = 2 \) (must be equation)  
   (ii) substituting \( x = 2 \) into \( f(x) \)  
   \[ y = 8 \]  

3. (a) \( q = -2, r = 4 \) or \( q = 4, r = -2 \)  
   (b) \( x = 1 \) (must be an equation)  
   (c) substituting \( (0, -4) \) into the equation  
   \[ -4 = p(0 - (-2))(0 - 4), -4 = p(-4)(2) \]  
   correct working towards solution  
   \[ p = \frac{4}{8} = \frac{1}{2} \]  

4. (a) evidence of attempting to solve \( f(x) = 0 \)  
   evidence of correct working  
   \[ (x+1)(x-2), \frac{1 \pm \sqrt{6}}{2} \]  
   intercepts are \((-1, 0) \) and \((2, 0) \) (accept \( x = -1, x = 2 \))  
   (b) evidence of appropriate method  
   \[ x_v = \frac{x_1 + x_2}{2}, x_v = -\frac{b}{2a}, \text{reference to symmetry} \]  
   \( x_v = 0.5 \)
5. (a) For a reasonable attempt to complete the square, (or expanding) (M1)
e.g. \(3x^2 - 12x + 11 = 3(x^2 - 4x + 4) + 11 - 12\)
f\((x) = 3(x - 2)^2 - 1\) (accept \(h = 2, k = 1\)) A1A1 N3

(b) **METHOD 1**
Vertex shifted to \((2 + 3, -1 + 5) = (5, 4)\) M1
so the new function is \(3(x - 5)^2 + 4\) (accept \(p = 5, q = 4\)) A1A1 N2

**METHOD 2**
g\((x) = 3((x - 3) - h)^2 + k + 5 = 3((x - 3) - 2)^2 - 1 + 5\)
= \(3(x - 5)^2 + 4\) (accept \(p = 5, q = 4\)) A1A1 N2

6. (a) **METHOD 1**
Using the discriminant = 0 \((q^2 - 4(4)(25) = 0\) ) M1
\(q^2 = 400\)
\(q = 20, q = -20\) A1A1 N2

**METHOD 2**
Using factorizing:
\((2x - 5)(2x - 5)\) and/or \((2x + 5) (2x + 5)\) M1
\(q = 20, q = -20\) A1A1 N2

(b) \(x = 2.5\) A1 N1

(c) \((0, 25)\) A1A1 N2

7. (a) (i) \(h = 3\) A1 N1

(ii) \(k = 1\) A1 N1

(b) \(g (x) = f (x - 3) + 1, 5 - (x - 3)^2 + 1, 6 - (x - 3)^2, -x^2 + 6x - 3\) A2 N2

(c)
8. (a) For attempting to complete the square or expanding \( y = 2(x - c)^2 + d \), or for showing the vertex is at (3, 5) \( \text{M1} \)

\[ y = 2(x - 3)^2 + 5 \quad \text{(accept } c = 3, \ d = 5) \quad \text{A1A1 N2} \]

(b) (i) \( k = 2 \) \( \text{A1 N1} \)

(ii) \( p = 3 \) \( \text{A1 N1} \)

(iii) \( q = 5 \) \( \text{A1 N1} \)

[6]

9. (a) (i) \( m = 3 \) \( \text{A2 N2} \)

(ii) \( p = 2 \) \( \text{A2 N2} \)

(b) Appropriate substitution \( \text{M1} \)

\[ eg \ 0 = d(1 - 3)^2 + 2, \ 0 = d(5 - 3)^2 + 2, \ 2 = d(3 - 1)(3 - 5) \]

\[ d = -\frac{1}{2} \quad \text{A1 N1} \]

[6]

10. (a) For a reasonable attempt to complete the square, (or expanding)

\[ 3x^2 - 12x + 11 = 3(x^2 - 4x + 4) + 11 - 12 \]

\[ = 3(x - 2)^2 - 1 \quad \text{(Accept } h = 2, \ k = 1) \quad \text{A1A1 2} \]

(b) METHOD 1

Vertex shifted to (2 + 3, –1 + 5) = (5, 4) \( \text{M1} \)

so the new function is \( 3(x - 5)^2 + 4 \) (Accept \( p = 5, \ q = 4 \)) \( \text{A1A1 2} \)

METHOD 2

\[ g(x) = 3((x - 3) - h)^2 + k + 5 = 3((x - 3) - 2)^2 - 1 + 5 \]

\[ = 3(x - 5)^2 + 4 \quad \text{(Accept } p = 5, \ q = 4) \quad \text{A1A1 2} \]

[6]

11. (a) \( p = -1 \) and \( q = 3 \) (or \( p = 3, \ q = -1 \)) \( \text{(A1)(A1) (C2)} \)

\( (accept \ (x + 1)(x - 3)) \)

(b) EITHER

by symmetry \( \text{(M1)} \)

OR

differentiating \( \frac{dy}{dx} = 2x - 2 = 0 \) \( \text{(M1)} \)

OR

Completing the square \( \text{(M1)} \)

\[ x^2 + 2x - 3 = x^2 - 2x + 1 - 4 = (x - 1)^2 - 4 \]

THEN

\( x = 1, \ y = -4 \quad \text{(so } C \text{ is } (1, -4)) \quad \text{(A1)(A1)(C2)(C1)} \)

(c) \( -3 \) \( \text{(A1) (C1)} \)

\( (accept \ (0, -3)) \)

[6]
12. Discriminant $\Delta = b^2 - 4ac = (-2k)^2 - 4$  
$\Delta > 0$  
$(2k)^2 - 4 > 0 \Rightarrow 4k^2 - 4 > 0$

EITHER  
$4k^2 > 4 \ (k^2 > 1)$  
OR  
$4(k-1)(k+1) > 0$  
OR  
$(2k-2)(2k+2) > 0$

THEN  
$k < -1$ or $k > 1$

13. One solution $\Rightarrow$ discriminant $= 0$  
$3^2 - 4k = 0$  
$9 = 4k$  
$k = \frac{9}{4} = 2\frac{1}{4}, 2.25$

14. (a) (i) $h = -1$  
(ii) $k = 2$

(b) $a(1+1)^2 + 2 = 0$  
$a = -0.5$

15. (a) $2x^2 - 8x + 5 = 2(x^2 - 4x + 4) + 5 - 8$  
$= 2(x-2)^2 - 3$  
$\Rightarrow a = 2, \ p = 2, \ q = -3$

(b) Minimum value of $2(x-2)^2 = 0$ (or minimum value occurs when $x = 2$)  
$\Rightarrow$ Minimum value of $f(x) = -3$  
$\Rightarrow$ Minimum value occurs at $(2, -3)$

16. (a) $2x^2 - 8x + 5 = 2(x^2 - 4x + 4) + 5 - 8$  
$= 2(x-2)^2 - 3$  
$\Rightarrow a = 2, \ p = 2, \ q = -3$

(b) Minimum value of $2(x-2)^2 = 0$ (or minimum value occurs when $x = 2$)  
$\Rightarrow$ Minimum value of $f(x) = -3$  
$\Rightarrow$ Minimum value occurs at $(2, -3)$
17. \[ y = (x+2)(x-3) \] \[ = x^2 - x - 6 \] Therefore, \( 0 = 4 - 2p + q \) (M1) (A1) (A1)(A1)(C2)(C2) OR \[ y = x^2 - x - 6 \] (C3) OR \[ 0 = 4 - 2p + q \] (A1) \[ 0 = 9 + 3p + q \] (A1) \[ p = -1, q = -6 \] (A1)(A1)(C2)(C2) [4]

18. \[ y = x^2 - x - 6 \] (C3) OR \[ 0 = 4 - 2p + q \] (A1) \[ 0 = 9 + 3p + q \] (A1) \[ p = -1, q = -6 \] (A1)(A1)(C2)(C2) [4]

19. (a) \[ f(x) = x^2 - 6x + 14 \] \[ f(x) = x^2 - 6x + 9 - 9 + 14 \] (M1) \[ f(x) = (x - 3)^2 + 5 \] (M1) \[ \] \[ \] \[ \] \[ \] \[ \] (A1)(A1) [4]

(b) Vertex is (3, 5) (A1)(A1) [4]

20. \[ 4x^2 + 4kx + 9 = 0 \] Only one solution \( \Rightarrow b^2 - 4ac = 0 \) (M1) \[ 16k^2 - 4(4)(9) = 0 \] (A1) \[ k^2 = 9 \] \[ k = \pm 3 \] (A1) \[ But given k > 0, k = 3 \] (A1) (C4) \[ OR \] \[ One solution \( \Rightarrow (4x^2 + 4kx + 9) \) is a perfect square \] (M1) \[ 4x^2 + 4kx + 9 = (2x \pm 3)^2 \] by inspection (A2) \[ given k > 0, k = 3 \] (A1) (C4) [4]

21. Graph of quadratic function.

<table>
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<tr>
<th>Expression</th>
<th>+</th>
<th>-</th>
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</tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>( b )</td>
<td></td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>( b^2 - 4ac )</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( b )</td>
<td>✓</td>
<td></td>
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</tr>
</tbody>
</table>

22. (a) \( x^2 - 3x - 10 = (x - 5)(x + 2) \)  
     (M1)(A1) (C2) 
(b) \( x^2 - 3x - 10 = 0 \Rightarrow (x - 5)(x + 2) = 0 \)  
     \( \Rightarrow x = 5 \) or \( x = -2 \)  
     (A1) (C2) [4] 

23. (a) \( p = -\frac{1}{2}, q = 2 \) or vice versa  
     (A1)(A1) (C2) 
(b) By symmetry \( C \) is midway between \( p, q \)  
     (M1)  
     \( \Rightarrow x \)-coordinate is \( \frac{-\frac{1}{2} + 2}{2} = \frac{3}{4} \)  
     (A1) (C2) [4] 

24. \( (7 - x)(1 + x) = 0 \)  
    \( \Leftrightarrow x = 7 \) or \( x = -1 \)  
     (M1)  
    \( B: x = \frac{7 + (-1)}{2} = 3; \)  
     (A1)  
    \( y = (7 - 3)(1 + 3) = 16 \)  
    (A1) (C2) [4]
2.4 Quadratic Functions

1. (a) Evidence of completing the square
   \[ f(x) = 2(x^2 - 6x + 9) + 5 - 18 \]
   \[ = 2(x - 3)^2 - 13 \] (accept \( h = 3, k = 13 \))  A1 N3

(b) Vertex is \((3, -13)\) A1 A1 N2

(c) \( x = 3 \) (must be an equation) A1 N1

(d) Evidence of using fact that \( x = 0 \) at \( y \)-intercept (M1)
   \( y \)-intercept is \((0, 5)\) (accept 5) A1 N2

(e) **METHOD 1**
   Evidence of using \( y = 0 \) at \( x \)-intercept (M1)
   e.g. \( 2(x - 3)^2 - 13 = 0 \)
   Evidence of solving this equation (M1)
   e.g. \( (x - 3)^2 = \frac{13}{2} \)

\[
(x - 3) = \pm \sqrt{\frac{13}{2}}
\]

\[
x = 3 \pm \sqrt{\frac{13}{2}} = 3 \pm \frac{\sqrt{26}}{2}
\]

\[
x = \frac{6 \pm \sqrt{26}}{2}
\]

\( p = 6, q = 26, r = 2 \) A1 A1 A1 N4

**METHOD 2**
Evidence of using \( y = 0 \) at \( x \)-intercept (M1)
E.g. \( 2x^2 - 12x + 5 = 0 \)
Evidence of using the quadratic formula (M1)
\[
x = \frac{12 \pm \sqrt{12^2 - 4 \times 2 \times 5}}{2 \times 2}
\]

\[
x = \frac{12 \pm \sqrt{104}}{4} \left( = \frac{6 \pm \sqrt{26}}{2} \right)
\]

\( p = 12, q = 104, r = 4 \) (or \( p = 6, q = 26, r = 2 \)) A1 A1 A1 N4

2. (a) \( f(x) = 3(x^2 + 2x + 1) - 12 \)
   \( = 3x^2 + 6x + 3 - 12 \) A1
   \( = 3x^2 + 6x - 9 \) A1 AG N0

(b) (i) Vertex is \((-1, -12)\) A1 A1 N2

(ii) \( x = -1 \) **must** be an equation A1 N1

(iii) \((0, -9)\) A1 N1

(iv) Evidence of solving \( f(x) = 0 \) (M1)
   E.g. factorizing, formula, correct working A1

\[
e.g. 3(x + 3)(x - 1) = 0, \ x = \frac{-6 \pm \sqrt{36 + 108}}{6}
\]

\((-3, 0), (1, 0)\) A1 A1 A1 N1
Notes: Award A1 for a parabola opening upward, A1 for vertex and intercepts in approximately correct positions.

(d) \( p = \begin{pmatrix} -1 \\ -12 \end{pmatrix} , \ t = 3 \) (accept \( p = -1, q = -12, t = 3 \)) A1A1A1 N3

3. (a) Vertex is (4, 8) A1A1 N2
(b) Substituting \(-10 = a(7-4)^2 + 8\)
\[ a = -2 \] A1 N1
(c) For \( y \)-intercept, \( x = 0 \) (A1)
\[ y = -24 \] A1 N2

4. (a) \((1, -2)\) A1A1 N2 2
(b) \( g(x) = 3(x-1)^2 - 2 \) (accept \( p = 1, q = -2 \)) A1A1 N2 2
(c) \((1, 2)\) A1A1 N2 2

5. (a) attempt to use discriminant (M1)
correct substitution, \((k-3)^2 - 4 \times k \times 1\) (A1)
setting their discriminant equal to zero M1
e.g. \((k-3)^2 - 4 \times k \times 1 = 0, k^2 - 10k + 9 = 0\)
\[ k = 1, k = 9 \] A1A1 N3
(b) \( k = 1, k = 9 \) A2 N2

6. (a) evidence of obtaining the vertex (M1)
e.g. a graph, \( x = -\frac{b}{2a} \), completing the square
\[ f(x) = 2(x+1)^2 - 8 \] A2 N3
(b) \( x = -1 \) (equation must be seen) A1 N1
(c) \( f(x) = 2(x-1)(x+3) \) A1A1 N2
### 2.5 Logarithmic and Exponential Functions

#### Paper 1

1. (a) attempt to apply rules of logarithms
   
   \( \text{e.g. } \ln a^b = b \ln a \), \( \ln ab = \ln a + \ln b \)
   
   correct application of \( \ln a^b = b \ln a \) (seen anywhere) \( \text{A1} \)
   
   \( \text{e.g. } 3 \ln x = \ln x^3 \)
   
   correct application of \( \ln ab = \ln a + \ln b \) (seen anywhere) \( \text{A1} \)
   
   \( \text{e.g. } \ln 5x^3 = \ln 5 + \ln x^3 \)
   
   so \( \ln 5x^3 = \ln 5 + 3 \ln x \)
   
   \( g(x) = f(x) + \ln 5 \) (accept \( g(x) = 3 \ln x + \ln 5 \)) \( \text{A1 N1} \)
   
   (b) transformation with correct name, direction, and value
   
   \( \text{e.g. translation by } \left( \begin{array}{c} 0 \\ \ln 5 \end{array} \right) \), shift up by \( \ln 5 \), vertical translation of \( \ln 5 \) \( \text{A3} \)

2. (a) \( f^{-1}(x) = \ln x \) \( \text{A1 N1} \)
   
   (b) (i) Attempt to form composite \( (f \circ g)(x) = f(\ln (1 + 2x)) \)
   
   \( (f \circ g)(x) = e^{\ln (1 + 2x)} = 1 + 2x \) \( \text{A1 N2} \)
   
   (ii) Simplifying \( y = e^{\ln (1 + 2x)} \) to \( y = 1 + 2x \) (may be seen in part (i) or later)
   
   Interchanging \( x \) and \( y \) (may happen any time) \( \text{M1} \)
   
   \( \text{e.g. } x = 1 + 2y \quad x - 1 = 2y \)
   
   \( (f \circ g)^{-1}(x) = \frac{x - 1}{2} \) \( \text{A1 N2} \)

3. (a) (i) \( f(a) = 1 \) \( \text{A1 N1} \)
   
   (ii) \( f(1) = 0 \) \( \text{A1 N1} \)
   
   (iii) \( f(a^4) = 4 \) \( \text{A1 N1} \)
   
   (b)

   ![Graph of a function and its inverse]

   \( \text{A1 A1 A1 N3} \)
4. (a) **METHOD 1**

\[ s^4 + 1 = s^4 \]  
\[ x + 1 = 4 \]  
\[ x = 3 \]  

**METHOD 2**

Taking logs  

\[ \text{eg } x + 1 = \log_5 625, (x + 1)\log 5 = \log 625 \]

\[ x + 1 = \frac{\log 625}{\log 5} \]  
\[ x + 1 = 4 \]  
\[ x = 3 \]

(b) **METHOD 1**

Attempt to re-arrange equation

\[ 3x + 5 = a^2 \]  
\[ x = \frac{a^2 - 5}{3} \]

**METHOD 2**

Change base to give \( \log (3x + 5) = \log a^2 \)

\[ 3x + 5 = a^2 \]  
\[ x = \frac{a^2 - 5}{3} \]

5. (a) **C** has equation \( x = 2^y \)

\[ \text{ie } y = \log_2 x \]

**OR**

Equation of **B** is \( x = \log_2 y \)

Therefore equation of **C** is \( y = \log_2 x \)

(b) Cuts \( x \)-axis \( \Rightarrow \log_2 x = 0 \)

\[ x = 2^0 \]
\[ x = 1 \]

Point is \((1, 0)\)
2.5 Logarithmic and Exponential Functions

1. (a) $253250$ (accept 253000)  
   Evidence of any appropriate approach  
   Correct substitution $250000 \times 1.013^{30}$  
   $368000$ (accept 368318)  

2. (a) $x = -1, (-1, 0), -1$  
   (b) (i) $f(-1.999) = \ln (0.001) = -6.91$  
   (ii) All real numbers.  
   (c) $(4.64, 1.89)$  

3. (a) $1 = A_0 e^{5k}$  
   Attempt to find $\frac{dA}{dr}$  
   Correct equation $0.2 = k \cdot A_0 e^{5k}$  
   For any valid attempt to solve the system of equations  
   $k = 0.2$  
   (b) $100 = \frac{1}{e} e^{0.2t}$  
   $t = \frac{\ln 100 + 1}{0.2} (= 28.0)$  

4. (a) $e^{\ln(x + 2)} = e^3$  
   $x + 2 = e^3$  
   $x = e^3 - 2 (= 18.1)$  
   (b) $\log_{10} (10^{2.5}) = \log_{10} 500$ (accept $\log$ and $\log_{10}$)  
   $2x = \log_{10} 500$  
   $x = \frac{1}{2} \log_{10} 500$  
   $x = 1.35$
5. \[10000e^{-0.3t} = 1500\]  
For taking logarithms  
\[-0.3\ln e = \ln 0.15\]  
\[t = \frac{\ln 0.15}{-0.3} = 6.32\]  
7 (years)  
\[\text{(A1)(C6)}\]  

6.  
(a) \[p = 100e^t\]  
\[= 100\]  
\[\text{(M1)(A1)(C2)}\]  
(b) Rate of increase is \[\frac{dp}{dt}\]  
\[\frac{dp}{dt} = 0.05 \times 100e^{0.05t} = 5e^{0.05t}\]  
When \(t = 10\)  
\[\frac{dp}{dt} = 5e^{0.05(10)} = 5e^{0.5} = 8.24 = 5\sqrt{e}\]  
\[\text{(A1)(C4)}\]  

7.  
(a) Initial mass \(\Rightarrow\) \(t = 0\)  
\[\text{mass} = 4\]  
\[\text{(A1)(C2)}\]  
(b) \[1.5 = 4e^{-0.2t}\] (or \(0.375 = e^{-0.2t}\))  
\[\ln 0.375 = -0.2t\]  
\[t = 4.90\text{ hours}\]  
\[\text{(M2)(A1)(C4)}\]  

8. \[15\%\text{ per annum} = \frac{15}{12}\% = 1.25\%\text{ per month}\]  
Total value of investment after \(n\) months, \[1000(1.0125)^n > 3000\]  
\[\Rightarrow (1.0125)^n > 3\]  
\[n \log (1.0125) > \log (3) \Rightarrow n > \frac{\log (3)}{\log (1.0125)}\]  
Whole number of months required so \(n = 89\) months.  
\[\text{(M1)(A1)(C6)}\]  

9.  
(a) \[\frac{15.2}{1.027} = 14.8\text{ million}\]  
\[\text{(M1)(A1)(C2)}\]  
(b) \[\frac{15.2}{(1.027)^3} = 13.3\text{ million}\]  
\[\text{(M1)(A1)(C2)}\]  

OR  
\[\frac{14.8}{(1.027)^4} = 13.3\text{ million}\]  
\[\text{(M1)(A1)(C2)}\]  

[4]
10. (a) At $t = 2$, $N = 10e^{0.4t}$
   $N = 22.3$ (3 sf)
   Number of leopards = 22 (A1)

(b) If $N = 100$, then solve $100 = 100e^{0.4t}$
   $10 = e^{0.4t}$
   $\ln 10 = 0.4t$
   $t = \frac{\ln 10}{0.4} \approx 5.76$ years (3 sf) (A1)

11. $1.023^t = 2$ (M1)
    $\Rightarrow t = \frac{\ln 2}{\ln 1.023}$ (M1)(A1)
    $= 30.48...$
    30 minutes (nearest minute) (A1) (C4)

12. (a) $n = 800e^0$
    $n = 800$ (A1)
    n' = $800$ A1 N2

(b) evidence of using the derivative
    $n'(15) = 731$ (M1)
    A1 N2

(c) **METHOD 1**
   setting up inequality (accept equation or reverse inequality) A1
   e.g. $n'(t) > 10000$
   evidence of appropriate approach M1
   e.g. sketch, finding derivative
   $k = 35.1226...$ (A1)
   least value of $k$ is 36 A1 N2

**METHOD 2**
   $n'(35) = 9842$, and $n'(36) = 11208$ A2
   least value of $k$ is 36 A2 N2

13. (a) combining 2 terms (A1)
    e.g. $\log_3 8x - \log_3 4$, $\log_3 \frac{1}{2} x + \log_3 4$
    expression which clearly leads to answer given A1
    e.g. $\log_3 \frac{8x}{3}$, $\log_3 \frac{4x}{2}$
    $f(x) = \log_3 2x$ AG N0 2

(b) attempt to substitute either value into $f$ (M1)
    e.g. $\log_3 1$, $\log_3 9$
    $f(0.5) = 0, f(4.5) = 2$ A1 A1 N3 3
(c) (i) $a = 2, b = 3$

(ii) $x = 0$ (must be an equation)

(d) $f^{-1}(0) = 0.5$

(e)
14. (a) (i) \( n = 5 \) 

\[ T = 280 \times 1.12^5 \]

\[ T = 493 \]

(ii) evidence of doubling (A1)

\[ e.g. 560 \]

setting up equation A1

\[ 280 \times 1.12^n = 560, 1.12^n = 2 \]

\( n = 6.116\ldots \) in the year 2007 A1 N3

(b) (i) \( P = \frac{2560000}{10 + 90e^{-0.1(5)}} \)

\[ P = 39635.993\ldots \] (A1)

\[ P = 39636 \]

(ii) \( P = \frac{2560000}{10 + 90e^{-0.1(7)}} \)

\[ P = 46806.997\ldots \]

not doubled A1 N0

valid reason for their answer R1

\( e.g. P < 51200 \)

(c) (i) correct value A2 N2

\( e.g. \frac{25600}{280}, 91.4, 640:7 \)

(ii) setting up an inequality (accept an equation, or reversed inequality) M1

\[ e.g. \frac{P}{T} < 70, \frac{2560000}{\left(10 + 90e^{-0.1n}\right)280 \times 1.12^n} < 70 \]

finding the value 9.31... (A1)

after 10 years A1 N2

15. (a) (i) 2420 (A1)

(ii) \( 1420 + 100n > 2000 \)

\( n > 5.8 \)

1999 (accept 6th year or \( n = 6 \)) (A1) (N1) 3

(b) (i) \( 1200000(1.025)^{10} = 1536101 \)

(accept 1 540 000 or 1.54(million)) (A1)

(ii) \( \frac{1536101 - 1200000}{1200000} \times 100 \)

28.0% (accept 28.3% from 1 540 000) (A1) (N2)

(iii) \( 1200000(1.025)^n > 2000000 \) (accept an equation) (M1)

\[ n \log 1.025 > \log \frac{2}{1.2} \Rightarrow n > 20.69 \] (M1)(A1)

2014 (accept 21st year or \( n = 21 \)) (A1) (N3) 7
16. (a) \[
\text{Value} = 1500(1.0525)^t
\]
\[
= 1748.87
\]
\[
= 1749 \text{ (nearest franc)}
\]
(b) \[
3000 = 1500(1.0525)^t \Rightarrow 2 = 1.0525^t
\]
\[
t = \frac{\log 2}{\log 1.0525} = 13.546
\]
It takes 14 years.
(c) \[
3000 = 1500(1 + r)^{10} \quad \text{or} \quad 2(1 + r)^{10}
\]
\[
\Rightarrow \frac{10}{\sqrt{2}} = 1 + r \quad \text{or} \quad \log 2 = 10 \log (1 + r)
\]
\[
\Rightarrow r = \frac{10}{\sqrt{2}} - 1 \quad \text{or} \quad r = 10^{\frac{\log 2}{10}} - 1
\]
\[
r = 0.0718 \quad [\text{or } 7.18\%]
\]

17. **Note:** A reminder that a candidate is penalized only once in this question for not giving answers to 3 sf

(a) \[
V(5) = 10000 \times (0.933)^5 = 7069.8 \ldots
\]
\[
= 7070 \text{ (3 sf)}
\]
(b) We want \( t \) when \( V = 5000 \)
\[
5000 = 10000 \times (0.933)^t
\]
\[
0.5 = 0.933^t
\]
\[
\frac{\log (0.5)}{\log (0.933)} = t \quad \text{or} \quad \frac{\ln (0.5)}{\ln (0.933)}
\]
\[
9.9949 = t
\]
After 10 minutes 0 seconds, to nearest second (or 600 seconds).
(c) \[0.05 = 0.933^t\]
\[
\frac{\log (0.05)}{\log (0.933)} = t = 43.197 \text{ minutes}
\]
\[\approx 3/4 \text{ hour}\]
(d) \[
10000 - 10000(0.933)^{0.001} = 0.693
\]
2.6 Solving Equations

1. recognizing \( \log a + \log b = \log ab \) (seen anywhere) (A1)
   e.g. \( \log_2(x(x - 2)), x^2 - 2x \)
   recognizing \( \log a, b = x \iff a^x = b \) (seen anywhere) (A1)
   e.g. \( 2^3 = 8 \)
   correct simplification (\( x(x - 2) = 2^3, x^2 - 2x - 8 \) (A1)
   evidence of correct approach to solve (M1)
   e.g. factorizing, quadratic formula
   correct working (A1)
   e.g. \((x - 4)(x + 2), \frac{2 \pm \sqrt{36}}{2}\)
   \( x = 4 \) (A2 N3)

2. \( e^{2x}(\sqrt{3} \sin x + \cos x) = 0 \) (A1)
   \( e^{2x} = 0 \) not possible (seen anywhere) (A1)
   simplifying (e.g. \( \sqrt{3} \sin x + \cos x = 0, \sqrt{3} \sin x = -\cos x, \frac{\sin x}{-\cos x} = \frac{1}{\sqrt{3}} \)) (A1)

   EITHER

   \( \tan x = -\frac{1}{\sqrt{3}} \) (A1)
   \( x = \frac{5\pi}{6} \) (A2 N4)

   OR

   sketch of 30°, 60°, 90° triangle with sides 1, 2, \( \sqrt{3} \) (A1)
   work leading to \( x = \frac{5\pi}{6} \) (A1)
   verifying \( \frac{5\pi}{6} \) satisfies equation (A1 N4)
3. (a) \( x^2 = 49 \)
   \[ x = \pm 7 \]  
   \[ x = 7 \]  
   \[ \text{(M1)} \]  
   \[ \text{A1} \]  
   \[ \text{N3} \]  

(b) \( 2^x = 8 \)
   \[ x = 3 \]  
   \[ \text{(M1)} \]  
   \[ \text{A1} \]  
   \[ \text{N2} \]  

(c) \[ x = 25^{\frac{1}{2}} \]
   \[ x = \frac{1}{\sqrt{25}} \]
   \[ x = \frac{1}{5} \]  
   \[ \text{A1} \]  
   \[ \text{N3} \]  

(d) \[ \log_2 (x(x - 7)) = 3 \]
   \[ \log_2 (x^2 - 7x) = 3 \]
   \[ 2^3 = 8 \quad (8 = x^2 - 7x) \]  
   \[ \text{(A1)} \]  
   \[ \text{A1} \]  
   \[ \text{(A1)} \]  
   \[ \text{(A1)} \]  
   \[ \text{(A1)} \]  
   \[ \text{N3} \]  

4. **METHOD 1**

\[
\log_9 81 + \log_9 \left(\frac{1}{9}\right) + \log_9 3 = 2 - 1 + \frac{1}{2} \]  
\[ \Rightarrow \frac{3}{2} = \log_9 x \]  
\[ \Rightarrow x = 9^{\frac{3}{2}} \]  
\[ \Rightarrow x = 27 \]  
\[ \text{(M1)} \]  
\[ \text{(A1)} \]  
\[ \text{(C4)} \]  

**METHOD 2**

\[
\log 81 + \log \left(\frac{1}{9}\right) + \log 3 = \log \left[ 81 \left(\frac{1}{9}\right)^3 \right] \]  
\[ = \log_9 27 \]  
\[ \Rightarrow x = 27 \]  
\[ \text{(A1)} \]  
\[ \text{(A1)} \]  
\[ \text{(C4)} \]  
[4]
1. (a) Two correct factors  
   \(eg\ y^2 + y - 12 = (y + 4)(y - 3), (2^x)^2 + (2^x) - 12 = (2^x + 4)(2^x - 3)\)  
   \(a = 4, b = -3\) (or \(a = -3, b = 4\))  
   \(N2\)

   \(b)\ 2^x - 3 = 0\)  
   \(2^x = 3\)  
   \(x = \log_2 3\)  
   \(A1\) \(N2\)

   **EITHER**

   Considering \(2^x + 4 = 0\) \(\left(2^x = -4\right)\) (may be seen earlier)  
   \(A1\)

   **Valid reason**  
   \(R1\) \(N1\)

   \(eg\) this equation has no real solution, \(2^x > 0\), graph does not cross the \(x\)-axis

   **OR**

   Considering graph of \(y = 22^x + 2^x - 12\) (asymptote does not need to be indicated)  
   \(A1\)

   ![Graph of \(y = 22^x + 2^x - 12\)]

   There is only one point of intersection of the graph with \(x\)-axis.  
   \(R1\) \(N1\)

2. \(\log_{27} (x(x - 0.4)) = 1\)  
   \(x^2 - 0.4x = 27\)  
   \(x = 5.4\) or \(x = -5\)  
   \(x = 5.4\)

   \(A1\) \(C6\)

   \(G2\)

   \(M1\) \(A1\)

   \(M1\)

   \([6]\)
3. **METHOD 1**
Using gdc equation solver for
\[ e^x + 2x - 5 = 0, \]  
\[ x = 1.0587 \]  
\[ = 1.059 \text{ (4 sf)} \]

**METHOD 2**
Using gdc to graph \( y = e^x \) and \( y = 5 - 2x \) and find x-coordinate at point of intersection.

\[ x = 1.0587 \]  
\[ = 1.059 \text{ (4 sf)} \]

4. (a)

(b) \( x = 0.876726 \text{ (6 sf)} \)
5. (a) 

Note: Award A1 for approximately correct shape, A1 for left end point in circle, A1 for local maximum in circle, A1 for right end point in circle.

(b) attempting to solve $g(x) = -1$

\[ \frac{1}{2} x \sin x + 1 = 0 \]

\[ x = 3.71 \quad \text{(A1 N2 2)} \quad [6] \]

6. evidence of appropriate approach

\[ e.g. \text{a sketch, writing } e^x - 4 \sin x = 0 \]

\[ x = 0.371, x = 1.36 \quad \text{(A2A2 N2N2)} \quad [5] \]

7. (a) 

(b) evidence of attempt to solve $f(x) = 1$

\[ e.g. \text{line on sketch, using } \tan x = \frac{\sin x}{\cos x} \]

\[ x = -0.207, x = 0.772 \quad \text{(A1A1 N3)} \quad [6] \]
8. (a) correct substitution
   \[ e.g. \ 25 + 16 - 40 \cos x, \ 5^2 + 4^2 - 2 \times 4 \times 5 \cos x \]
   \[ AC = \sqrt{41 - 40 \cos x} \]
   AG

(b) correct substitution
   \[ e.g. \ \frac{AC}{\sin x} = \frac{4}{\sin 30}, \ \frac{1}{2} AC = 4 \sin x \]
   \[ AC = 8 \sin x \left( \frac{4\sin x}{\sin 30} \right) \]
   A1 N1

(c) (i) evidence of appropriate approach using AC
   \[ e.g. \ 8 \sin x = \sqrt{41 - 40 \cos x}, \ sketch \ showing \ intersection \]
   correct solution 8.682..., 111.317...
   obtuse value 111.317...
   (A1)
   (A1)
   \[ x = 111.32 \ to \ 2 \ dp \ (do \ not \ accept \ the \ radian \ answer \ 1.94) \]
   A1 N2

(ii) substituting value of \( x \) into either expression for AC
   \[ e.g. \ AC = 8 \sin 111.32 \]
   AC = 7.45
   A1 N2

(d) (i) evidence of choosing cosine rule
   \[ e.g. \ \cos B = \frac{a^2 + c^2 - b^2}{2ac} \]
   correct substitution
   \[ e.g. \ \frac{4^2 + 4^2 - 7.45^2}{2 \times 4 \times 4}, \ 7.45^2 = 32 - 32 \cos y, \ \cos y = -0.734... \]
   \[ y = 137 \]
   A1 N2

(ii) correct substitution into area formula
   \[ e.g. \ \frac{1}{2} \times 4 \times 4 \times \sin 137, \ 8 \sin 137 \]
   area = 5.42
   A1 N2
9. (a) using the cosine rule \( a^2 = b^2 + c^2 - 2bc \cos \hat{A} \)  
substituting correctly \( BC^2 = 65^2 + 104^2 - 2(65)(104)\cos60° \)  
\[ \Rightarrow BC = 91m \]  
A1 N2

(b) finding the area, using \( \frac{1}{2} bc \sin \hat{A} \)  
substituting correctly, area = \( \frac{1}{2} (65)(104)\sin60° \)  
\[ = 1690\sqrt{3} \] (accept \( p = 1690 \))  
A1 N2

(c) (i) \[ A_1 = \left(\frac{1}{2}\right)(65)(x)\sin30° \]  
\[ = \frac{65x}{4} \]  
AG N0

(ii) \[ A_2 = \left(\frac{1}{2}\right)(104)(x)\sin30° \]  
\[ = 26x \]  
A1 N1

(iii) stating \( A_1 + A_2 = A \) or substituting \( \frac{65x}{4} + 26x = 1690\sqrt{3} \)  
\[ \Rightarrow x = 40\sqrt{3} \] (accept \( q = 40 \))  
A1 N2

(d) (i) Recognizing that supplementary angles have equal sines  
\[ e.g. \: \hat{A}DC = 180° - \hat{A}DB \Rightarrow \sin\hat{A}DC = \sin\hat{A}DB \]  
R1

(ii) using sin rule in \( \triangle ADB \) and \( \triangle ACD \)  
\[ \Rightarrow \frac{BD}{\sin30°} = \frac{65}{\sin\hat{A}DB} \Rightarrow BD = \frac{65}{\sin\hat{A}DB} \sin30° \]  
A1

and \[ \frac{DC}{\sin30°} = \frac{104}{\sin\hat{A}DC} \Rightarrow DC = \frac{104}{\sin\hat{A}DC} \sin30° \]  
M1

since \( \sin\hat{A}DB = \sin\hat{A}DC \)  
\[ \frac{BD}{65} = \frac{DC}{104} \Rightarrow \frac{BD}{DC} = \frac{65}{104} \]  
A1

\[ \Rightarrow \frac{BD}{DC} = \frac{5}{8} \]  
AG N0