Chapter 16
PROPERTIES OF CURVES

EXERCISE 16A

1. We seek the tangent to \( y = x - 2x^2 + 3 \) at \( x = 2 \).
   When \( x = 2 \), \( y = 2 - 2(2)^2 + 3 = -3 \).
   \( \therefore \) the point of contact is \((2, -3)\).
   Now \( \frac{dy}{dx} = 1 - 4x \), so at \( x = 2 \),
   \[ \frac{dy}{dx} = 1 - 8 = -7 \]
   \( \therefore \) the tangent has equation
   \[ \frac{y - (-3)}{x - 2} = -7 \]
   \( \therefore \) \( y + 3 = -7(x - 2) \)
   \[ \therefore \] \( y = -7x + 14 - 3 \)
   \[ \therefore \] \( y = -7x + 11 \)

2. We seek the tangent to \( y = x^3 - 5x \) at \( x = 1 \).
   When \( x = 1 \), \( y = 1^3 - 5(1) = -4 \).
   \( \therefore \) the point of contact is \((1, -4)\).
   Now \( \frac{dy}{dx} = 3x^2 - 5 \), so at \( x = 1 \),
   \[ \frac{dy}{dx} = 3 - 5 = -2 \]
   \( \therefore \) the tangent has equation
   \[ \frac{y - (-4)}{x - 1} = -2 \]
   \( \therefore \) \( y + 4 = -2x + 2 \)
   \[ \therefore \] \( y = -2x - 2 \)

3. We seek the tangent to \( y = \frac{3}{x} - \frac{1}{x^2} = 3x^{-1} - x^{-2} \) at \((-1, -4)\).
   Now \( \frac{dy}{dx} = -3x^{-2} + 2x^{-3} \)
   \( = -\frac{3}{x^2} + \frac{2}{x^3} \) so at \((-1, -4)\).
   \[ \frac{dy}{dx} = -\frac{3}{(-1)^2} + \frac{2}{(-1)^3} \]
   \( = -3 - 2 \)
   \( = -5 \)
   \( \therefore \) the tangent has equation
   \[ \frac{y - (-4)}{x - (-1)} = -5 \]
   \( \therefore \) \( y + 4 = -5x - 5 \)
   \[ \therefore \] \( y = -5x - 9 \)

b. We seek the tangent to \( y = \sqrt{x^2 + 1} = x^2 + 1 \) at \( x = 4 \).
   When \( x = 4 \), \( y = \sqrt{4^2 + 1} = 3 \).
   \( \therefore \) the point of contact is \((4, 3)\).
   Now \( \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \), so at \( x = 4 \),
   \[ \frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4} \]
   \( \therefore \) the tangent has equation
   \[ \frac{y - 3}{x - 4} = \frac{1}{4} \]
   \( \therefore \) \( 4y - 12 = x - 4 \)
   \[ \therefore \] \( 4y = x + 8 \)

d. We seek the tangent to \( y = \frac{4}{\sqrt{x}} \) at \((1, 4)\).
   Now \( y = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}} \).
   \( \therefore \) \( \frac{dy}{dx} = -2x^{-\frac{3}{2}} \) so at \( x = 1 \),
   \[ \frac{dy}{dx} = -2 \left(1^{-\frac{3}{2}}\right) = -2 \]
   \( \therefore \) the tangent has equation
   \[ \frac{y - 4}{x - 1} = -2 \]
   \( \therefore \) \( y - 4 = -2x + 2 \)
   \[ \therefore \] \( y = -2x + 6 \)

f. We seek the tangent to \( y = \frac{3x^2 - \frac{1}{x}}{x} = 3x^2 - x^{-1} \) at \( x = -1 \).
   When \( x = -1 \), \( y = 3(-1)^2 - (-1) = 4 \).
   \( \therefore \) the point of contact is \((-1, 4)\).
   Now \( \frac{dy}{dx} = 6x + x^{-2} \)
   \( = 6x + \frac{1}{x^2} \) so at \( x = -1 \),
   \[ \frac{dy}{dx} = 6(-1) + \frac{1}{(-1)^2} = -5 \]
   \( \therefore \) the tangent has equation
   \[ \frac{y - 4}{x - (-1)} = -5 \]
   \( \therefore \) \( y - 4 = -5x - 5 \)
   \[ \therefore \] \( y = -5x - 1 \)
2 a We seek the normal to \( y = x^2 \) at (3, 9).

Now \( \frac{dy}{dx} = 2x \) so at \( x = 3 \),
\[
\frac{dy}{dx} = 2(3) = 6 = \frac{1}{6}
\]
\[\therefore \text{the normal at (3, 9) has gradient } -\frac{1}{6}, \]
so the equation of the normal is
\[
\frac{y - 9}{x - 3} = -\frac{1}{6}
\]
\[\therefore 6y - 54 = -x + 3 \]
\[\therefore 6y = -x + 57 \]

b We seek the normal to \( y = x^3 - 5x + 2 \) at \( x = -2 \).

When \( x = -2 \), \( y = (-2)^3 - 5(-2) + 2 = 4 \)
\[\therefore \text{and the point of contact is } (-2, 4) \]
Now \( \frac{dy}{dx} = 3x^2 - 5 \) so at \( x = -2 \),
\[
\frac{dy}{dx} = 3(-2)^2 - 5 = 7
\]
\[\therefore \text{the normal at } (-2, 4) \text{ has gradient } -\frac{1}{7}, \]
so the equation of the normal is
\[
\frac{y - 4}{x - (-2)} = -\frac{1}{7}
\]
\[\therefore 7y - 28 = -(x + 2) \]
\[\therefore 7y = -x + 26 \]

c We seek the normal to \( y = \frac{5}{\sqrt{x}} - \sqrt{x} \) at (1, 4).

Now \( y = 5x^{-\frac{1}{2}} - x^{\frac{1}{2}} \)
\[\therefore \frac{dy}{dx} = -\frac{5}{2} x^{-\frac{3}{2}} - \frac{1}{2} x^{-\frac{1}{2}} \] so at \( x = 1 \),
\[
\frac{dy}{dx} = -\frac{5}{2} - \frac{1}{2} = -3
\]
\[\therefore \text{the normal at (1, 4) has gradient } \frac{1}{3}, \]
so the equation of the normal is
\[
\frac{y - 4}{x - 1} = \frac{1}{3}
\]
\[\therefore 3y - 12 = x - 1 \]
\[\therefore 3y = x + 11 \]

d We seek the normal to \( y = 8\sqrt{x} - \frac{1}{x^2} \) at \( x = 1 \).

When \( x = 1 \), \( y = 8\sqrt{1} - \frac{1}{1^2} = 7 \)
\[\therefore \text{the point of contact is } (1, 7) \]
Now \( y = 8\sqrt{x} - \frac{1}{x^2} = 8x^{\frac{1}{2}} - x^{-2} \)
\[\therefore \frac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3} \] so at \( x = 1 \),
\[
\frac{dy}{dx} = 4 + 2 = 6
\]
\[\therefore \text{the normal at (1, 7) has gradient } -\frac{1}{6}, \]
so the equation of the normal is
\[
\frac{y - 7}{x - 1} = -\frac{1}{6}
\]
\[\therefore 6y - 42 = -x + 1 \]
\[\therefore 6y = -x + 43 \]

3 a \[y = 2x^3 + 3x^2 - 12x + 1 \]
\[\therefore \frac{dy}{dx} = 6x^2 + 6x - 12 \]
Horizontal tangents have gradient = 0
so \( 6x^2 + 6x - 12 = 0 \)
\[\therefore x^2 + x - 2 = 0 \]
\[\therefore (x + 2)(x - 1) = 0 \]
\[\therefore x = -2 \text{ or } x = 1 \]
Now at \( x = -2 \),
\[y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 \]
\[= 21 \]
and at \( x = 1 \),
\[y = 2(1)^3 + 3(1)^2 - 12(1) + 1 \]
\[= -6 \]
\[\therefore \text{the points of contact are } (-2, 21) \text{ and } (1, -6) \]
\[\therefore \text{the tangents are } y = 21 \text{ and } y = -6 \]

b Now \( y = 2\sqrt{x} + \frac{1}{\sqrt{x}} = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} \)
\[\therefore \frac{dy}{dx} = x^{-\frac{1}{2}} - \frac{1}{2} x^{-\frac{3}{2}} = \frac{1}{\sqrt{x}} - \frac{1}{2x^{\frac{3}{2}}} \]
Horizontal tangents have gradient = 0
\[\therefore \frac{1}{\sqrt{x}} - \frac{1}{2x^{\frac{3}{2}}} = 0 \]
\[\therefore 2x - 1 = 0 \]
\[\therefore x = \frac{1}{2} \]
Now at \( x = \frac{1}{2} \),
\[y = 2\sqrt{\frac{1}{2}} + \frac{1}{\sqrt{\frac{1}{2}}} = \frac{2(\frac{1}{2}) + 1}{\sqrt{\frac{1}{2}}} = \frac{2}{\sqrt{\frac{1}{2}}} = 2\sqrt{2} \]
\[\therefore \text{the only horizontal tangent touches at the curve at } \left(\frac{1}{2}, 2\sqrt{2}\right) \]
Now \( y = 2x^3 + kx^2 - 3 \)

\[
\frac{dy}{dx} = 6x^2 + 2kx
\]

When \( x = 2 \), \( \frac{dy}{dx} = 4 \)

\[
6(2)^2 + 2k(2) = 4
\]

\[
24 + 4k = 4
\]

\[4k = -20\]

\[k = -5\]

Now \( y = 1 - 3x + 12x^2 - 8x^3 \)

\[
\frac{dy}{dx} = -3 + 24x - 24x^2
\]

When \( x = 1 \), \( \frac{dy}{dx} = -3 + 24 - 24 = -3 \)

\[\text{the tangent at (1, 2) has gradient } -3\]

The tangents to the curve have gradient \(-3\) when \(-3 + 24x - 24x^2 = -3\)

\[
24x^2 - 24x = 0
\]

\[
24x(x - 1) = 0
\]

\[\text{when } x = 0 \text{ or } x = 1\]

So the other \( x \)-value for which the tangent to the curve has gradient \(-3\) is \( x = 0 \), and when \( x = 0 \), \( y = 1 - 0 + 0 - 0 = 1 \)

\[\text{the tangent to the curve at (0, 1) is parallel to the tangent at (1, 2).}\]

This tangent has equation \[
\frac{y - 1}{x} = -3
\]

or \( y = -3x + 1 \).

Now \( y = \sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}} \)

\[
\frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}
\]

At \( x = 4 \), \( \frac{dy}{dx} = \frac{a}{2} \left( \frac{4}{2} \right) - \frac{b}{2} \left( \frac{1}{2} \right) \)

\[= \frac{a}{4} - \frac{b}{16}\]

\[\text{the gradient of the tangent to the curve at } x = 4 \text{ will be } \frac{a}{4} - \frac{b}{16} = \frac{4a - b}{16}\]

However the equation of the \textit{normal} is \( 4x + y = 22 \) or \( y = -4x + 22 \).

\[\text{the normal has gradient } -4\]

\[\text{the tangent has gradient } \frac{1}{4}, \text{ and so } \]

\[\frac{4a - b}{16} = \frac{1}{4}\]

\[4a - b = 4\]

\[b = 4a - 4 \] .... (1)

Also, at \( x = 4 \) the normal line intersects the curve.

\[a\sqrt{4} + \frac{b}{\sqrt{4}} = -4(4) + 22\]

\[2a + \frac{b}{2} = 6\]

Consequently, \( 2a + \frac{4a - 4}{2} = 6 \) \{using (1)\}

\[2a + 2a - 2 = 6\]

\[4a = 8\]

\[a = 2\]

and so \( b = 4(2) - 4 = 4 \) \{from (1)\}

\[
\begin{align*}
\frac{dy}{dx} &= 4x \\
\text{at the point where } x &= a, \quad \frac{dy}{dx} = 4a \\
\text{the gradient of the tangent at the point } x &= a \text{ is } 4a. \\
\text{Also, at } x &= a, \quad y = 2a^2 - 1. \\
\text{the tangent has equation } y &= (2a^2 - 1) \\
&= 4a \\
&= 2a^2 + 1 = 4a(x - a) \\
&= 2a^2 + 1 = 4ax - 4a^2 \\
&= 4ax - y = 2a^2 + 1
\end{align*}
\]
5 a \( y = \sqrt{2x + 1} \)

When \( x = 4 \), \( y = \sqrt{2(4) + 1} = 3 \), so the point of contact is \((4, 3)\).

Now \( \frac{dy}{dx} = \frac{1}{2}(2x + 1)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{2x + 1}} \)

\[ \therefore \text{at } x = 4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{2(4) + 1}} = \frac{1}{3} \]

\( \therefore \text{the tangent has equation } \frac{y - 3}{x - 4} = \frac{1}{3} \)

or \( 3y = x + 5 \).

b \( y = \frac{1}{2 - x} = (2 - x)^{-1} \)

\[ \therefore \text{at } x = -1, \quad y = \frac{1}{2 - (-1)} = \frac{1}{3} \]

So the point of contact is \((-1, \frac{1}{3})\).

Now \( \frac{dy}{dx} = -(2 - x)^{-2}(-1) = \frac{1}{(2 - x)^2} \)

\[ \therefore \text{at } x = -1, \quad \frac{dy}{dx} = \frac{1}{(2 - (-1))^2} = \frac{1}{9} \]

\( \therefore \text{the tangent has equation } \frac{y - \frac{1}{3}}{x - (-1)} = \frac{1}{9} \)

\[ \therefore 9y - 3 = x + 1 \]

\[ \therefore 9y = x + 4 \]

c We seek the tangent to \( f(x) = \frac{x}{1 - 3x} \) at \((-1, -\frac{1}{4})\).

\( f(x) \) is a quotient where \( u = x \) and \( v = 1 - 3x \)

\( \therefore u' = 1 \) and \( v' = -3 \)

Now \( f'(x) = \frac{u'v - uv'}{v^2} \) \{quotient rule\}

\[ \therefore f'(x) = \frac{1(1 - 3x) - x(-3)}{(1 - 3x)^2} = \frac{1}{(1 - 3x)^2} \]

\[ \therefore f'(-1) = \frac{1}{(1 - 3(-1))^2} = \frac{1}{16} \]

\( \therefore \text{the tangent has equation } \frac{y - (-\frac{1}{4})}{x - (-1)} = \frac{1}{16} \)

\[ \therefore 16y + 4 = x + 1 \]

\[ \therefore 16y = x - 3 \]

d We seek the tangent to \( f(x) = \frac{x^2}{1 - x} \) at \((2, -4)\).

\( f(x) \) is a quotient where \( u = x^2 \) and \( v = 1 - x \)

\( \therefore u' = 2x \) and \( v' = -1 \)

Now \( f'(x) = \frac{u'v - uv'}{v^2} \) \{quotient rule\}

\[ \therefore f'(x) = \frac{2x(x - 1) - x^2(-1)}{(1 - x)^2} = \frac{2x - 2x^2 + x^2}{(1 - x)^2} = \frac{2x - x^2}{(1 - x)^2} \]

\[ \therefore f'(2) = \frac{2(2) - 4}{(1 - 2)^2} = \frac{0}{1} = 0 \]

As the tangent has gradient 0, it is horizontal.

\( \therefore \text{its equation is } y = c \)

Since the contact point is \((2, -4)\), the tangent has equation \( y = -4 \).

6 a We seek the normal to \( y = \frac{1}{(x^2 + 1)^2} \) at \((1, \frac{1}{4})\).

As \( y = (x^2 + 1)^{-2} \),

\[ \frac{dy}{dx} = -2(x^2 + 1)^{-3}(2x) = \frac{-4x}{(x^2 + 1)^3} \]

\[ \therefore \text{at } x = 1, \quad \frac{dy}{dx} = \frac{-4}{(1 + 1)^3} = -\frac{4}{8} = -\frac{1}{2} \]

\( \therefore \text{the normal at } (1, \frac{1}{4}) \) has gradient 2.

So the equation of the normal is

\[ \frac{y - \frac{1}{4}}{x - 1} = 2 \]

\[ \therefore y - \frac{1}{4} = 2x - 2 \]

\[ \therefore y = 2x - \frac{7}{4} \]

b \( y = \frac{1}{\sqrt{3 - 2x}} \)

\[ \therefore \text{at } x = -3, \quad y = \frac{1}{\sqrt{3 - 2(-3)}} = \frac{1}{3} \]

\( \therefore \text{the point of contact is } (-3, \frac{1}{3}) \)

Now \( y = (3 - 2x)^{-\frac{1}{2}} \)

\[ \frac{dy}{dx} = -\frac{1}{2}(3 - 2x)^{-\frac{3}{2}}(-2) = (3 - 2x)^{-\frac{3}{2}} \]

\[ \therefore \text{at } x = -3, \quad \frac{dy}{dx} = (3 - 2(-3))^{-\frac{3}{2}} = 9^{-\frac{3}{2}} = 3^{-3} = \frac{1}{27} \]

\( \therefore \text{the normal at } (-3, \frac{1}{3}) \) has gradient \(-27\).

So the equation of the normal is

\[ \frac{y - \frac{1}{3}}{x - (-3)} = -27 \]

\[ \therefore y - \frac{1}{3} = -27(x + 3) \]

\[ \therefore y = -27x - \frac{82}{3} \]
c \[ f(x) = \sqrt{x}(1 - x)^2 \]
Since \[ f(4) = \sqrt{4}(1 - 4)^2 = 18, \]
the point of contact is \((4, 18)\).
Now \(f(x)\) is a product where
\[ u = x^{\frac{1}{2}} \quad \text{and} \quad v = (1 - x)^2 \]
∴ \[ u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2(1 - x) \]
\[ = -2(1 - x) \]
Now \(f'(x) = u'v + uv'\) \{product rule\}
\[ \therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1 - x)^2 - x^{\frac{1}{2}}2(1 - x) \]
\[ = \frac{1}{2}(9) - 2(2)(-3) = \frac{57}{4} \]
∴ the normal at \((4, 18)\) has gradient \(-\frac{4}{57}\).
So, the equation of the normal is
\[ \frac{y - 18}{x - 4} = -\frac{4}{57} \]
∴ \[ 57(y - 18) = -4(x - 4) \]
∴ \[ 57y = -4x + 1042 \]

\[ \text{d} \]
\[ f(x) = \frac{x^2 - 1}{2x + 3} \]
Since \[ f(-1) = \frac{(-1)^2 - 1}{2(-1) + 3} = \frac{0}{1} = 0 \]
the point of contact is \((-1, 0)\).
Now \(f(x)\) is a quotient where
\[ u = x^2 - 1 \quad \text{and} \quad v = 2x + 3 \]
∴ \[ u' = 2x \quad \text{and} \quad v' = 2 \]
Now \(f'(x) = \frac{uv' - vu'}{v^2}\)
\[ \therefore f'(x) = \frac{2x(2x + 3) - (x^2 - 1)(2)}{(2x + 3)^2} \]
\[ = \frac{2(-1)(-2 + 3) - ((-1)^2 - 1)(2)}{(2(-1) + 3)^2} \]
\[ = -2(1)^2 = -2 \]
∴ the normal at \((-1, 0)\) has gradient \(\frac{1}{2}\).
So, the equation of the normal is
\[ \frac{y - 0}{x - (-1)} = \frac{1}{2} \]
or \[ 2y = x + 1 \]

7 The tangent has equation \(3x + y = 5\) or \(y = -3x + 5\)
∴ the tangent has gradient \(-3\) \(\ldots (1)\)
Also, at \(x = -1, y = -3(-1) + 5 = 8\)
∴ the tangent contacts the curve at \((-1, 8)\) \(\ldots (2)\)
Now \(y = a(1 - bx)^{\frac{1}{2}}, \text{ so } \frac{dy}{dx} = \frac{1}{2}a(1 - bx)^{-\frac{1}{2}}(-b)\)
∴ \[\frac{3}{2} = \frac{1}{2}a(1 + b)^{-\frac{1}{2}}(-b) \quad \{\text{using (1)}\} \]
∴ \[6 = \frac{ab}{\sqrt{1 + b}} \quad \ldots (3)\]
Using (2), \((-1, 8)\) must lie on the curve \[y = a\sqrt{1 - bx} \quad \quad (4)\]
∴ \[\frac{6\sqrt{1 + b}}{b} = \frac{8}{\sqrt{1 + b}} \quad \{\text{equating as in (3) and (4)}\}\]
∴ \[6(1 + b) = 8b \]
∴ \[6 + 6b = 8b \]
∴ \[b = 3 \quad \text{and} \quad a = \frac{8}{\sqrt{4}} = 4 \]

8 \[a \]
\[f(x) = e^{-x}\]
∴ \[f(1) = e^{-1}\]
∴ the point of contact is \((1, \frac{1}{e})\).
Now \(f'(x) = -e^{-x}\)
∴ \[f'(1) = -e^{-1} = -\frac{1}{e}\]
So, the gradient of the tangent is \(-\frac{1}{e}\)
∴ the tangent has equation \[\frac{y - \frac{1}{e}}{x - 1} = -\frac{1}{e} \]
∴ \[e(y - \frac{1}{e}) = -(x - 1) \]
∴ \[ey - 1 = -x + 1 \]
∴ \[x + ey = 2 \quad \text{or} \quad y = -\frac{1}{e}x + \frac{2}{e} \]
\[ y = \ln(2-x) \]
so when \( x = -1 \), \( y = \ln 3 \)
\[ \therefore \text{the point of contact is } (-1, \ln 3). \]
\[ \therefore \text{the tangent has equation} \quad \frac{y - \ln 3}{x + 1} = -\frac{1}{3} \]
Now \[ \frac{dy}{dx} = \frac{-1}{2-x} \]
\[ \therefore \text{when } x = -1, \quad \frac{dy}{dx} = \frac{-1}{2-(-1)} = -\frac{1}{3} \]
\[ \therefore 3(y - \ln 3) = -(x + 1) \quad \therefore y - 3\ln 3 = -x - 1 \]
So, the gradient of the tangent is \(-\frac{1}{3}\).

ej \[ y = \ln \sqrt{x} \quad \therefore \text{when } y = -1, \quad -1 = \frac{1}{2} \ln x \]
\[ = \ln x^{\frac{1}{2}} \quad \therefore \ln x = -2 \]
\[ = \frac{1}{2} \ln x \quad \therefore x = e^{-2} \]
\[ \therefore x = \frac{1}{e^2} \quad \therefore \text{the point of contact is } \left( \frac{1}{e^2}, -1 \right) \]
Now \[ \frac{dy}{dx} = \frac{1}{2x} = \frac{1}{2e^{-2}} = \frac{e^2}{2} \]
\[ \therefore \text{the tangent has gradient } \frac{e^2}{2} \text{ and the normal has gradient } -\frac{2}{e^2} \]
\[ \therefore \text{the normal has equation} \quad \frac{y + 1}{x - \frac{1}{e^2}} = -\frac{2}{e^2} \]
\[ \therefore e^2(y + 1) = -2 \left( x - \frac{1}{e^2} \right) \]
\[ \therefore e^2y + e^2 = -2x + \frac{2}{e^2} \]
\[ \therefore 2x + e^2y = \frac{2}{e^2} - e^2 \quad \text{or} \quad y = -\frac{2}{e^2} x + \frac{2}{e^4} - 1 \]

\[ y = \frac{\cos x}{1 + \sin x} \quad \therefore \frac{dy}{dx} = \frac{(-\sin x)(1 + \sin x) - \cos x(\cos x)}{(1 + \sin x)^2} \]
\[ = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \]
\[ = \frac{-\sin x - \sin^2 x}{(1 + \sin x)^2} \quad \{ \sin^2 x + \cos^2 x = 1 \} \]
\[ = \frac{-(1 + \sin x)}{(1 + \sin x)^2} \]
\[ = \frac{-1}{1 + \sin x} \]
Since \( \frac{-1}{1 + \sin x} \) never equals 0, there are no horizontal tangents.

10 \( a \quad y = \sin x \quad \therefore \frac{dy}{dx} = \cos x \quad b \quad y = \tan x \quad \therefore \frac{dy}{dx} = \frac{1}{\cos^2 x} \)
\[ \text{When } x = 0, \quad \frac{dy}{dx} = \cos 0 = 1 \quad \text{When } x = 0, \quad \frac{dy}{dx} = \frac{1}{\cos^2 0} = 1 \]
\[ \therefore \text{the tangent has equation} \quad \frac{y - 0}{x - 0} = 1 \quad \therefore \text{the tangent has equation} \quad \frac{y - 0}{x - 0} = 1 \]
\[ \text{or } y = x \quad \text{or } y = x \]
\[ y = \cos x \quad \therefore \quad \frac{dy}{dx} = -\sin x \]

When \( x = \frac{\pi}{6} \), \( y = \frac{\sqrt{3}}{2} \)

and \( \frac{dy}{dx} = -\sin \left( \frac{\pi}{6} \right) = -\frac{1}{2} \)

So, the normal has gradient 2,

and its equation is \( \frac{y - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{6}} = 2 \)

\[ \therefore \quad y - \frac{\sqrt{3}}{2} = 2x - \frac{\pi}{3} \]

\[ \therefore \quad 2x - y = \frac{\pi}{3} - \frac{\sqrt{3}}{2} \]

\[ \text{Consider the tangent to } y = x^3 \text{ at } x = 2. \]

When \( x = 2 \), \( y = 2^3 = 8 \) so the point of contact is \((2, 8)\).

Now \( \frac{dy}{dx} = 3x^2 \) and so at \( x = 2 \),

\[ \frac{dy}{dx} = 3(2)^2 = 12 \]

\[ \therefore \quad \text{the tangent at } (2, 8) \text{ has gradient 12 and} \]

its equation is \( \frac{y - 8}{x - 2} = 12 \)

\[ \therefore \quad y - 8 = 12x - 24 \]

\[ \therefore \quad y = 12x - 16 \]

\[ \therefore \quad \text{the tangent meets the curve where} \]

\[ 12x - 16 = x^3 \]

\[ \therefore \quad x^3 - 12x + 16 = 0 \]

Because the tangent touches the curve at \( x = 2 \), there must be a repeated solution at this point.

\[ (x - 2)^2 \] must be a factor of this cubic

\[ (x - 2)(x + 4) = 0 \]

\[ \therefore \quad \text{the tangent meets the curve again when} \]

\( x = -4 \).

When \( x = -4 \), \( y = (-4)^3 = -64 \)

\[ \therefore \quad \text{the tangent meets the curve again at} \]

\( (-4, -64) \).

\[ f(x) = x^2 + \frac{4}{x^3} \]

\[ \therefore \quad f'(x) = 2x - 2 \times \frac{4}{x^3} \]

\[ \therefore \quad f''(x) = 2x - \frac{8}{x^3} \]

\[ \text{Horizontal tangents have gradient 0, so} \]

\[ 2x - \frac{8}{x^3} = 0 \]

\[ \therefore \quad 2x^4 = 8 \]

\[ \therefore \quad x^4 = 4 \]

\[ \therefore \quad x = \pm \sqrt{2} \]

\[ y = \frac{1}{\sin(2x)} = (\sin(2x))^{-1} \]

\[ \therefore \quad \frac{dy}{dx} = -1(\sin(2x))^{-2} \times 2 \cos(2x) \]

\[ = -\frac{2 \cos(2x)}{(\sin(2x))^2} \]

When \( x = \frac{\pi}{4}, y = 1 \)

and \( \frac{dy}{dx} = -\frac{2 \cos(\frac{\pi}{4})}{(\sin(\frac{\pi}{4}))^2} = 0 \)

\[ \therefore \quad \text{the gradient of the normal is undefined, so the normal is} \]

\( x = \frac{\pi}{2} \).

\[ \text{Consider the tangent to} \]

\[ y = -x^3 + 2x^2 + 1 \text{ at } x = -1. \]

When \( x = -1 \), \( y = -(1)^3 + 2(-1)^2 + 1 = 4 \)

and so the point of contact is \((-1, 4)\).

Now \( \frac{dy}{dx} = -3x^2 + 4x \) and so at \( x = -1 \),

\[ \frac{dy}{dx} = -3(-1)^2 + 4(-1) = -7 \]

\[ \therefore \quad \text{the tangent at } (-1, 4) \text{ has gradient } -7 \]

and its equation is \( \frac{y - 4}{x - (-1)} = -7 \)

\[ \therefore \quad y - 4 = -7(x + 1) \]

\[ \therefore \quad y = -7x - 3 \]

\[ \therefore \quad \text{the tangent meets the curve where} \]

\[ -7x - 3 = -x^3 + 2x^2 + 1 \]

\[ \therefore \quad x^3 - 2x^2 - 7x - 4 = 0 \]

Because the tangent touches the curve at \( x = -1 \), there must be a repeated solution at this point.

\[ (x + 1)^2 \] must be a factor of this cubic

\[ (x + 1)(x - 4) = 0 \]

\[ \therefore \quad \text{the tangent meets the curve again when} \]

\( x = 4 \).

When \( x = 4 \), \( y = -(4)^3 + 2(4)^2 + 1 = -64 + 32 + 1 = -31 \)

\[ \therefore \quad \text{the tangent meets the curve again at} \]

\( (4, -31) \).

\[ f(x) = x^2 + \frac{4}{x^3} \]

\[ \therefore \quad f'(-\sqrt{2}) = (-\sqrt{2})^2 + \frac{4}{(-\sqrt{2})^3} = 2 + \frac{4}{2} = 4 \]

\[ \therefore \quad \text{the horizontal tangent at } (-\sqrt{2}, 4) \text{ is} \]

\( y = 4. \)

When \( x = \sqrt{2}, \)

\[ f(\sqrt{2}) = (\sqrt{2})^2 + \frac{4}{(\sqrt{2})^3} = 2 + \frac{4}{2} = 4 \]

\[ \therefore \quad \text{the horizontal tangent at } (\sqrt{2}, 4) \text{ is} \]

\( y = 4. \)

\[ \therefore \quad \text{the tangents are the same line because they have the same equation.} \]
13 \[ y = x^2e^x \] so when \( x = 1 \), \( y = e \)

\[ \therefore \text{ the point of contact is } (1, e). \]

\[ \text{The tangent cuts the } x\text{-axis when } y = 0 \]
\[ \therefore 3ex = 2e \]
\[ \therefore x = \frac{2}{3} \]

\[ \text{and the } y\text{-axis when } x = 0 \]
\[ \therefore y = 2e \]
\[ \therefore y = -2e \]
\[ \therefore 3ex - y = 2e \]

So, \( A \) is \( \left( \frac{2}{3}, 0 \right) \) and \( B \) is \( (0, -2e) \).

14 a Consider the tangent to \( y = x^2 - x + 9 \) at \( x = a \).

When \( x = a \), \( y = a^2 - a + 9 \), so the point of contact is \( (a, a^2 - a + 9) \).

\[ \text{Now } \frac{dy}{dx} = 2x - 1 \text{ and so at } x = a, \frac{dy}{dx} = 2a - 1 \]

\[ \therefore \text{the gradient of the tangent at } (a, a^2 - a + 9) \text{ is } 2a - 1 \]

\[ \therefore \text{the equation of the tangent is } \frac{y - (a^2 - a + 9)}{x - a} = 2a - 1 \]

\[ \therefore y - (a^2 - a + 9) = (2a - 1)(x - a) \]
\[ \therefore y = (2a - 1)x - 2a^2 + a + a^2 - a + 9 \]
\[ \therefore y = (2a - 1)x - a^2 + 9 \] .... (1)

But this tangent passes through \((0, 0)\), so \(0 = a^2 - 9\)

\[ \therefore (a + 3)(a - 3) = 0 \]
\[ \therefore a = \pm 3 \]

\[ \therefore \text{the tangents are: } \text{At } a = 3, \ y = (2(3) - 1)x - 3^2 + 9 \{\text{from (1)}\} \]
\[ \therefore y = 5x, \text{ with contact at } (3, 15). \]

At \( a = -3 \), \( y = (2(-3) - 1)x - (-3)^2 + 9 \{\text{from (1)}\} \]
\[ \therefore y = -7x, \text{ with contact at } (-3, 21). \]

b Let \( (a, a^3) \) lie on \( y = x^3 \).

\[ \text{Now } \frac{dy}{dx} = 3x^2, \text{ so at } x = a, \frac{dy}{dx} = 3a^2 \]

\[ \therefore \text{the gradient of the tangent at } (a, a^3) \text{ is } 3a^2 \]

\[ \therefore \text{the equation of the tangent is } \frac{y - a^3}{x - a} = 3a^2 \text{ or } y - a^3 = (3a^2)(x - a) \]

But this tangent passes through \((-2, 0)\), so \(0 - a^3 = 3a^2(-2 - a)\)

\[ \therefore -a^3 = -6a^2 - 3a^3 \]
\[ \therefore 2a^3 + 6a^2 = 0 \]
\[ \therefore 2a^2(a + 3) = 0 \]
\[ \therefore a = 0 \text{ or } -3 \]

If \( a = 0 \), the tangent equation is \( y = 0 \), with contact point \((0, 0)\).

If \( a = -3 \), the tangent equation is \( y - (-27) = 27(x + 3) \)
\[ \therefore y = 27x + 54, \text{ with contact point } (-3, -27). \]

c Let \( (a, \sqrt{a}) \) lie on \( y = \sqrt{x} \).

\[ \text{Now } \frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{a}}, \text{ so at } x = a, \frac{dy}{dx} = \frac{1}{2\sqrt{a}} \]

\[ \therefore \text{the gradient of the tangent at } (a, \sqrt{a}) \text{ is } \frac{1}{2\sqrt{a}} \]

and the gradient of the normal at this point is \(-2\sqrt{a}\).
\[ \frac{y - \sqrt{a}}{x - a} = -2\sqrt{a} \]

or \[ y - \sqrt{a} = -2\sqrt{a}(x - a). \]

But this normal passes through \((4, 0)\), so \[ 0 - \sqrt{a} = -2\sqrt{a}(4 - a) \]
\[ \therefore 2\sqrt{a}(4 - a) - \sqrt{a} = 0 \]
\[ \therefore \sqrt{a}(8 - 2a - 1) = 0 \]
\[ \therefore \sqrt{a}(7 - 2a) = 0 \]
\[ \therefore a = 0 \text{ or } \frac{7}{2}. \]

But \(a = 0\) is the endpoint of the function, so there is no normal here.

When \(a = \frac{7}{2}\), \[ y - \sqrt{\frac{7}{2}} = -2\sqrt{\frac{7}{2}}(x - \frac{7}{2}) \]
\[ \therefore \sqrt{2}y - \sqrt{7} = -2\sqrt{7}(x - \frac{7}{2}) \]
\[ \therefore \sqrt{2}y + 2\sqrt{7}x = 7\sqrt{7} + \sqrt{7} \]
\[ \therefore \sqrt{2}y + 2\sqrt{7}x = 8\sqrt{7} \]
\[ \therefore y = -\sqrt{14}x + 4\sqrt{14} \]

with contact point \((\frac{7}{2}, \sqrt{\frac{7}{2}})\).

15 \( y = e^x \) so when \( x = a \), \( y = e^a \)
\[ \therefore \text{the point of contact is } (a, e^a). \]

Now \( \frac{dy}{dx} = e^x \)
\[ \therefore \text{at the point } (a, e^a), \frac{dy}{dx} = e^a \]
\[ \therefore \text{the tangent has equation } \frac{y - e^a}{x - a} = e^a \]
\[ \therefore y - e^a = e^a(x - a) \quad \text{... (1)} \]
Since the tangent passes through the origin, \((0, 0)\) must satisfy \((1)\)
\[ \therefore 0 - e^a = e^a(0 - a) \]
\[ \therefore -e^a = -ae^a \]
\[ \therefore e^a(a - 1) = 0 \]
\[ \therefore a = 1 \quad \text{(as } e^a > 0) \]
So the equation of the tangent is \( y - e = ex - e \) or \( y = ex \).

16 \( a \)

\[ f(x) = \frac{8}{x^2} \]
\[ f'(x) = -16x^{-3} = -\frac{16}{x^3} \]
\[ f'(a) = -\frac{16}{a^3} \]
\[ \therefore \text{the gradient of the tangent at } (a, \frac{8}{a^2}) \text{ is } -\frac{16}{a^3} \]
\[ \therefore \text{the equation of the tangent is } \frac{y - \frac{8}{a^2}}{x - a} = -\frac{16}{a^3} \]
\[ \therefore a^3y - 8a = -16x + 16a \]
\[ \therefore 16x + a^3y = 24a \]

4 The tangent cuts the \(x\)-axis when \( y = 0 \)
\[ \therefore 16x = 24a \]
\[ \therefore x = \frac{3}{2}a \]
\[ \therefore \text{A is } \left(\frac{3}{2}a, 0\right). \]

The tangent cuts the \(y\)-axis when \( x = 0 \)
\[ \therefore a^3y = 24a \]
\[ \therefore y = \frac{24}{a^3} \]
\[ \therefore \text{B is } \left(0, \frac{24}{a^2}\right). \]

\[ \text{Area of triangle OAB} \]
\[ = \frac{1}{2} \times \left(\frac{3}{2}a\right) \times \left(\frac{24}{a^2}\right) \]
\[ = \frac{18}{|a|} \text{ units}^2 \]
\[ \text{As } a \to \infty, \frac{18}{a} \to 0 \]
\[ \therefore \text{area } \to 0 \]
17 \[ y = 3e^{-x} \text{ and } y = 2 + e^x \text{ meet when } 3e^{-x} = 2 + e^x \]
\[ \therefore 3 = 2e^x + e^2x \quad \{ \times e^x \} \]
\[ \therefore e^{2x} + 2e^x - 3 = 0 \]
\[ \therefore (e^x + 3)(e^x - 1) = 0 \]
\[ \therefore e^x = -3 \text{ or } 1 \]
\[ \therefore e^x = 1 \text{ and so } x = 0 \quad \{ \text{as } e^x > 0 \} \]

Now when \( x = 0, \ y = 3e^0 = 3, \) so the graphs meet at \((0, 3)\).

For \( y = 2 + e^x, \ \frac{dy}{dx} = e^x, \)
so at the point \((0, 3), \ \frac{dy}{dx} = e^0 = 1 \)

\[ \therefore \text{the gradient of the tangent at this point is } 1 \]

\[ \therefore \text{the tangent has direction vector } \left\langle \frac{1}{1} \right\rangle \]

For \( y = 3e^{-x}, \ \frac{dy}{dx} = -3e^{-x}, \)
so at the point \((0, 3), \ \frac{dy}{dx} = -3 \)

\[ \therefore \text{the gradient of the tangent at this point is } -3 \]

\[ \therefore \text{the tangent has direction vector } \left\langle \frac{-3}{1} \right\rangle \]

If \( \theta \) is the acute angle between the tangents, then
\[ \cos \theta = \frac{|1(1) + 1(-3)|}{\sqrt{1^2 + 1^2} \sqrt{1^2 + (-3)^2}} = \frac{|-2|}{2\sqrt{10}} = \frac{2}{\sqrt{20}} \]
\[ \therefore \theta \approx 63.43^\circ \]

18 a \( y = ax^2, \ a > 0 \) touches \( y = \ln x \) when \( ax^2 = \ln x \)

If the curves touch when \( x = b \) then \( ab^2 = \ln b \) \( \ldots (1) \)

Now for \( y = ax^2, \ \frac{dy}{dx} = 2ax \) and for \( y = \ln x, \ \frac{dy}{dx} = \frac{1}{x} \)
\[ \therefore \text{when } x = b, \ \frac{dy}{dx} = 2ab \]
\[ \therefore \text{when } x = b, \ \frac{dy}{dx} = \frac{1}{b} \]

Since the curves touch each other, they share a common tangent. \( \therefore \frac{1}{b} = 2ab \) \( \ldots (2) \)

b Now \( ab^2 = \frac{1}{2} \) \( \{ \text{from (2)} \} \)

and \( ab^2 = \ln b \) \( \{ \text{from (1)} \} \)
\[ \therefore \ln b = \frac{1}{2} \]
\[ \therefore b = e^{\frac{1}{2}} = \sqrt{e} \]

When \( x = b = \sqrt{e}, \ y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2} \)
\[ \therefore \text{the point of contact is } \left( \sqrt{e}, \frac{1}{2} \right) \]

c \( a = \frac{1}{2b^2} \) \( \{ \text{from (2)} \} \)
\[ \therefore a = \frac{1}{2(\sqrt{e})^2} = \frac{1}{2e} \]

d The tangent has gradient \( \frac{1}{b} = \frac{1}{\sqrt{e}} \) and passes through \( \left( \sqrt{e}, \frac{1}{2} \right) \)
\[ \therefore \text{the tangent is } \frac{y - \frac{1}{2}}{x - \sqrt{e}} = \frac{1}{\sqrt{e}} \therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}} \left( x - \sqrt{e} \right) \]
\[ \therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}} x - 1 \]
\[ \therefore y = e^{-\frac{1}{2}} x - \frac{1}{2} \]

**EXERCISE 16B**

1 a

\[ y \]

\[ x \]

\[ (3, -3) \]

\[ x \geq 0 \quad \text{II never} \]

b

\[ y \]

\[ (-2, 2) \]

\[ (3, -3) \]

\[ x \]

\[ x \leq 2 \quad \text{II } x \geq 2 \]

c

\[ y \]

\[ (2, 3) \]

\[ x \]

\[ x \leq 2 \quad \text{II } x \geq 2 \]
2. a. \( f(x) = x^2, \quad f'(x) = 2x \)
   
   Sign diagram of \( f'(x) \):
   
   Increasing when \( x \geq 0 \),
   Decreasing when \( x \leq 0 \)

   b. \( f(x) = -x^3, \quad f'(x) = -3x^2 \)
   
   Sign diagram of \( f'(x) \):
   
   Decreasing for all \( x \)

   c. \( f(x) = 2x^2 + 3x - 4, \quad f'(x) = 4x + 3 \)
   
   Sign diagram of \( f'(x) \):
   
   Increasing when \( x \geq -\frac{3}{4} \),
   Decreasing when \( x \leq -\frac{3}{4} \)

   d. \( f(x) = \sqrt{x} = x^{\frac{1}{2}} \),
   
   \( f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \)
   
   Sign diagram of \( f'(x) \):
   
   \( f(x) \) is only defined when \( x \geq 0 \)
   Increasing when \( x \geq 0 \), never decreasing

   e. \( f(x) = e^x \), \( f'(x) = e^x \)
   
   Sign diagram of \( f'(x) \):
   
   \( f(x) \) is increasing for all \( x \)

   f. \( f(x) = x^3 - 6x^2, \quad f'(x) = 3x^2 - 12x = 3x(x-4) \)
   
   Sign diagram of \( f'(x) \):
   
   Increasing when \( x \leq 0 \) or \( x \geq 4 \),
   Decreasing when \( 0 \leq x \leq 4 \)

   g. \( f(x) = \ln x, \quad f'(x) = \frac{1}{x} \)
   
   Sign diagram of \( f'(x) \):
   
   \( f(x) \) is only defined when \( x > 0 \)
   Increasing when \( x > 0 \), never decreasing

   h. \( f(x) = -4x^3 + 15x^2 + 18x + 3 \)
   
   \( f'(x) = -12x^2 + 30x + 18 \)
   
   \( = -6(2x^2 - 5x - 3) \)
   
   \( = -6(2x+1)(x-3) \)
   
   Sign diagram of \( f'(x) \):
   
   Increasing when \( -\frac{1}{2} \leq x \leq 3 \),
   Decreasing when \( x \leq -\frac{1}{2} \) or \( x \geq 3 \)
\( f(x) = 3 + e^{-x} \), \( f'(x) = -e^{-x} \)

Sign diagram of \( f'(x) \):

\( f(x) \) is decreasing for all \( x \).

\( f(x) = 3x^4 - 16x^3 + 24x^2 - 2, \)

\( f'(x) = 12x^3 - 48x^2 + 48x = 12x(x^2 - 4x + 4) = 12x(x - 2)^2 \)

Sign diagram of \( f'(x) \):

increasing when \( x \geq 0 \),
decreasing when \( x \leq 0 \)

\( f(x) = x^3 - 6x^2 + 3x - 1, \)

\( f'(x) = 3x^2 - 12x + 3 = 3(x^2 - 4x + 1) \)

\( f'(x) = 0 \) when \( x = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3} \)

Sign diagram of \( f'(x) \):

increasing when \( x \leq 2 - \sqrt{3} \)
or \( x \geq 2 + \sqrt{3} \),
decreasing when \( 2 - \sqrt{3} \leq x \leq 2 + \sqrt{3} \)

\( f(x) = \frac{4x}{x^2 + 1} \)
is a quotient with \( u = 4x \) and \( u = x^2 + 1 \)

\( u' = 4 \) and \( u' = 2x \)

\( f'(x) = \frac{4(x^3 + 1) - 4x \times 2x}{(x^2 + 1)^2} = \frac{4x^2 + 4 - 8x^2}{(x^2 + 1)^2} = \frac{4 - 4x^2}{(x^2 + 1)^2} = \frac{-4(x^2 - 1)}{(x^2 + 1)^2} = \frac{-4(x + 1)(x - 1)}{(x^2 + 1)^2} \)

Sign diagram of \( f'(x) \):

\( f(x) \) is increasing for \(-1 \leq x \leq 1\),
decreasing for \( x \leq -1 \) and \( x \geq 1 \)

\( f(x) = xe^x, \)

\( f'(x) = e^x + xe^x = e^x(1 + x) \)

Sign diagram of \( f'(x) \):

increasing when \( x \geq -1 \)
decreasing when \( x \leq -1 \)

\( f(x) = 2x^3 + 9x^2 + 6x - 7, \)

\( f'(x) = 6x^2 + 18x + 6 = 6(x^2 + 3x + 1) \)

\( f'(x) = 0 \) when \( x = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2} \)

Sign diagram of \( f'(x) \):

increasing for \( x \leq 2 \sqrt{5} \) or \( x \geq -3 + \sqrt{5} \),
decreasing for \( \frac{-3 - \sqrt{5}}{2} \leq x \leq \frac{-3 + \sqrt{5}}{2} \)

\( f(x) = x - 2\sqrt{x} = x - 2x^{\frac{1}{2}} \)

\( f'(x) = 1 - x^{-\frac{1}{2}} = 1 - \frac{1}{\sqrt{x}} = \frac{\sqrt{x} - 1}{\sqrt{x}} \)

Sign diagram of \( f'(x) \):

increasing when \( x \geq 1 \),
decreasing when \( 0 \leq x \leq 1 \)

\( f(x) = \frac{4x}{(x - 1)^2} \)
is a quotient with \( u = 4x \) and \( u = (x - 1)^2 \)

\( u' = 4 \) and \( u' = 2(x - 1) \)

\( f'(x) = \frac{4(x - 1)^2 - 8x(x - 1)}{(x - 1)^4} = \frac{4(x - 1)[(x - 1) - 2x]}{(x - 1)^4} = \frac{4(-1 - x)}{(x - 1)^3} = \frac{-4(x - 1)}{(x - 1)^3} \)

Sign diagram of \( f'(x) \):

\( f(x) \) is increasing for \(-1 \leq x < 1 \),
decreasing for \( x \leq -1 \) and \( x > 1 \)
5 a \( f(x) = \frac{-x^2 + 4x - 7}{x - 1} \) is a quotient with \( u = -x^2 + 4x - 7 \) and \( v = x - 1 \)
\[ \therefore \quad u' = -2x + 4 \quad \text{and} \quad v' = 1 \]
\[ \therefore \quad f'(x) = \frac{(-2x + 4)(x - 1) - (-x^2 + 4x - 7)(1)}{(x - 1)^2} \]
\[ = \frac{-2x^2 + 6x - 4 + x^2 - 4x + 7}{(x - 1)^2} \]
\[ = \frac{-x^2 + 2x + 3}{(x - 1)^2} \]
\[ = \frac{-(x^2 - 2x - 3)}{(x - 1)^2} \]
\[ = \frac{-(x + 1)(x - 3)}{(x - 1)^2} \]

Sign diagram of \( f'(x) \):

\[ \begin{array}{cccc}
& -1 & + & \frac{1}{3} & - \\
\hline
\frac{1}{3} & 1 & & & \frac{1}{3} \\
\frac{1}{3} & & & & -1 \\
\frac{1}{3} & & & & -1 \\
\frac{1}{3} & & & & -1 \\
\end{array} \]

b \( f(x) \) is increasing for \(-1 \leq x < 1\) and \(1 < x \leq 3\), and decreasing for \(x \leq -1\) and \(x \geq 3\).

6 a \( f(x) = \frac{x^3}{x^2 - 1} \) is a quotient with \( u = x^3 \) and \( v = x^2 - 1 \)
\[ \therefore \quad u' = 3x^2 \quad \text{and} \quad v' = 2x \]
\[ \therefore \quad f'(x) = \frac{3x^2(x^2 - 1) - x^3 \times 2x}{(x^2 - 1)^2} \]
\[ = \frac{3x^4 - 3x^2 - 2x^4}{(x^2 - 1)^2} \]
\[ = \frac{x^2(x^2 - 3)}{(x^2 - 1)^2} \]
\[ = \frac{x^2(x + \sqrt{3})(x - \sqrt{3})}{(x^2 - 1)^2} \]

Sign diagram of \( f'(x) \):

\[ \begin{array}{cccccc}
& + & - & \frac{1}{\sqrt{3}} & - & - \\
\hline
\frac{1}{\sqrt{3}} & -1 & & & 0 & 1 \\
\frac{1}{\sqrt{3}} & & & & 1 & \sqrt{3} \\
\frac{1}{\sqrt{3}} & & & & -1 & -1 \\
\frac{1}{\sqrt{3}} & & & & -1 & -1 \\
\end{array} \]

b \( f(x) = e^{-x^2} \)

Sign diagram of \( f'(x) \):

\[ \begin{array}{cccc}
& + & - & 0 \\
\hline
0 & & & - \\
\end{array} \]

\[ \therefore \quad f(x) \) is increasing for \( x \leq 0 \) and decreasing for \( x \geq 0 \).

\[ f(x) = x^2 + \frac{4}{x - 1} = x^2 + 4(x - 1)^{-1} \]
\[ \therefore \quad f'(x) = 2x - 4(x - 1)^{-2} \times 1 \]
\[ = 2x - \frac{4}{(x - 1)^2} \]
\[ = \frac{2x(x - 1)^2 - 4}{(x - 1)^2} \]
\[ = \frac{2x(x^2 - 2x + 1) - 4}{(x - 1)^2} \]
\[ = \frac{2x^3 - 4x^2 + 2x - 4}{(x - 1)^2} \]
\[ = \frac{(x - 2)(2x^2 + 2)}{(x - 1)^2} \]

Sign diagram of \( f'(x) \):

\[ \begin{array}{cccc}
& - & \frac{1}{1} & + \\
\hline
\frac{1}{2} & & & - \\
\frac{1}{2} & & & - \\
\frac{1}{2} & & & - \\
\frac{1}{2} & & & - \\
\end{array} \]

\[ \therefore \quad f(x) \) is increasing for \( x \geq 2 \), and decreasing for \( x < 1 \) and \( 1 < x \leq 2 \).
\[ f(x) = \frac{e^{-x}}{x} \] is a quotient with \[ u = e^{-x} \] and \[ v = x \]

\[ u' = -e^{-x} \quad \text{and} \quad v' = 1 \]

\[ f'(x) = \frac{-e^{-x}x - e^{-x} \times 1}{x^2} = \frac{-e^{-x}(x + 1)}{x^2} \]

\[ \therefore f'(x) \text{ is increasing for } x \leq -1, \text{ and decreasing for } -1 \leq x < 0 \quad \text{and} \quad x > 0. \]

**EXERCISE 16C**

1. a. A is a local maximum, B is a stationary inflection, C is a local minimum.
   
   b. \( f'(x) \) has sign diagram:

   ![Sign Diagram](image)

   c. \( f(x) \) is increasing for \( x \leq -2 \) and \( x \geq 3 \)

   d. \( f(x) \) has sign diagram:

   ![Sign Diagram](image)

2. a. \( f(x) = x^2 - 2 \therefore f'(x) = 2x \)
   
   ![Graph](image)

   b. \( f(x) = x^3 + 1 \therefore f'(x) = 3x^2 \)
   
   ![Graph](image)

   c. \( f(x) = x^3 - 3x + 2 \)

   \[ f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1) \]

   ![Graph](image)

   d. \( f(x) = x^4 - 2x^3 \)

   \[ f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1) \]

   ![Graph](image)
\[ f(x) = x^3 - 6x^2 + 12x + 1 \]
\[ f'(x) = 3x^2 - 12x + 12 \]
\[ = 3(x^2 - 4x + 4) \]
\[ = 3(x - 2)^2 \]

with sign diagram:

Now \( f(2) = 9 \), so there is a stationary inflection at \((2, 9)\).

\[ f(x) = x - \sqrt{x} \]
\[ f'(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}} \]
\[ = 1 - \frac{1}{2\sqrt{x}} \]

with sign diagram:

\( f(x) \) is defined for all \( x \geq 0 \)

Now \( f\left(\frac{1}{4}\right) = -\frac{1}{4} \), so there is a local minimum at \(\left(\frac{1}{4}, -\frac{1}{4}\right)\).

\[ f(x) = 1 - x\sqrt{x} = 1 - x^{\frac{3}{2}} \]
\[ f'(x) = -\frac{3}{2}x^{\frac{1}{2}} = -\frac{3\sqrt{x}}{2} \]

with sign diagram:

\( f(x) \) is only defined when \( x \geq 0 \)

Now \( f(0) = 1 \), so there is a local maximum at \((0, 1)\).

\[ f(x) = \sqrt{x} + 2 \]
\[ f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \]
\[ = \frac{1}{2\sqrt{x}} \neq 0 \]

with sign diagram:

\( f(x) \) is not stationary.

\[ f(x) = x^4 - 6x^2 + 8x - 3 \]
\[ f'(x) = 4x^3 - 12x + 8 \]
\[ = 4(x^3 - 3x + 2) \]
\[ = 4(x - 1)(x^2 + x - 2) \]
\[ = 4(x - 1)(x + 2)(x - 1) \]

with sign diagram:

Now \( f(-2) = -27 \), \( f(1) = 0 \), so there is a local minimum at \((-2, -27)\), and a stationary inflection at \((1, 0)\).

\[ f(x) = x^4 - 2x^2 - 8 \]
\[ f'(x) = 4x^3 - 4x \]
\[ = 4x(x^2 - 1) = 4x(x + 1)(x - 1) \]

with sign diagram:

Now \( f(-1) = -9 \), \( f(1) = 0 \), \( f(0) = -8 \), so there are local minima at \((-1, -9)\) and \((1, -9)\), and a local maximum at \((0, -8)\).
3 \[ f(x) = ax^2 + bx + c, \quad a \neq 0 \]
\[ \therefore f'(x) = 2ax + b \]

\( f(x) \) has a stationary point when \( f'(x) = 0 \)
\[ \therefore x = -\frac{b}{2a} \]

There is a local maximum when \( a < 0 \) and there is a local minimum when \( a > 0 \)

4 \( a \)
\[ y = xe^{-x} \]
\[ \therefore \frac{dy}{dx} = 1e^{-x} - xe^{-x} \quad \{\text{product rule}\} \]
\[ = e^{-x}(1 - x) \]
\[ = \frac{1 - x}{e^x} \]

which has sign diagram:

When \( x = 1, \ y = 1e^{-1} = \frac{1}{e} \), so we have a local maximum at \( (1, \frac{1}{e}) \).

\( b \)
\[ y = xe^x \]
\[ \therefore \frac{dy}{dx} = 2xe^x + xe^x \quad \{\text{product rule}\} \]
\[ = xe^x(2 + x) \]

which has sign diagram:

When \( x = -2, \ y = 4e^{-2} \), and when \( x = 0, \ y = 0 \).

So, we have a local maximum at \( (-2, \frac{4}{e^2}) \), and a local minimum at \( (0, 0) \).

\( c \)
\[ y = \frac{e^x}{x} \]
\[ \therefore \frac{dy}{dx} = \frac{e^x x - e^x (1)}{x^2} \quad \{\text{quotient rule}\} \]
\[ = \frac{e^x(x - 1)}{x^2} \]

which has sign diagram:

When \( x = 1, \ y = \frac{e^1}{1} = e \), so we have a local minimum at \( (1, e) \).

\( d \)
\[ y = e^{-x}(x + 2) \]
\[ \therefore \frac{dy}{dx} = -e^{-x}(x + 2) + e^{-x} \quad \{\text{product rule}\} \]
\[ = e^{-x}(-x - 2 + 1) \]
\[ = e^{-x}(-x - 1) \]

which has sign diagram:

When \( x = -1, \ y = e(-1 + 2) = e \), so we have a local maximum at \( (-1, e) \).

5 \[ f(x) = 2x^2 + ax^2 - 24x + 1 \]
\[ \therefore f'(x) = 6x^2 + 2ax - 24 \]

But \( f'(-4) = 0 \), so \( 96 - 8a - 24 = 0 \)
\[ \therefore 72 = 8a \]
\[ \therefore a = 9 \]

6 \( a \)
\[ f(x) = ax^3 + bx + b \]
\[ \therefore f'(x) = 3ax^2 + a \]

But \( f'(-2) = 0 \)
\[ \therefore 3(-2)^2 + a = 0 \]
\[ \therefore 12 + a = 0 \]
\[ \therefore a = -12 \]

Also, \( f(-2) = 3 \)
\[ ( -2 )^3 - 12(-2) + b = 3 \]
\[ -8 + 24 + b = 3 \]
\[ \therefore b = -13 \]
Now \( f(x) = x^3 - 12x - 13 \)
\[ f'(x) = 3x^2 - 12 \]
\[ = 3(x^2 - 4) \]
\[ = 3(x + 2)(x - 2) \] with sign diagram:

Now \( f(2) = -29 \), so there is a local minimum at \((2, -29)\) and a local maximum at \((-2, 3)\).

7  a  \( f(x) \) is defined when \( \ln x \) is defined \( \therefore f(x) \) is defined for \( x > 0 \)

\( f'(x) = \ln x + \frac{x}{x} \) \{product rule\}
\[ = \ln x + 1 \]
which is 0 when \( \ln x = -1 \)
\[ \therefore x = e^{-1} \]

Sign diagram of \( f'(x) \) is:

So, there is a local minimum at \( \left( \frac{1}{e}, \frac{1}{e} \ln \frac{1}{e} \right) \)
\[ \therefore \text{the global minimum value of } f(x) \text{ is } \frac{1}{e} \ln e^{-1} = -\frac{1}{e} \]

8  a  If \( f(x) = \sin x \) then \( f'(x) = \cos x \)
Stationary points occur when \( f'(x) = 0 \),
which is when \( x = \frac{\pi}{2}, 2\frac{\pi}{2} \)
Sign diagram for \( f'(x) \) is:

There is a local maximum at \( \left( \frac{\pi}{2}, 1 \right) \)
and a local minimum at \( \left( 2\frac{\pi}{2}, -1 \right) \).

\( \text{If } f(x) = \cos (2x) \text{ then } f'(x) = -2 \sin (2x) \)
\[ \therefore f'(x) = 0 \text{ when } -2 \sin (2x) = 0 \]
\[ \therefore \sin (2x) = 0 \]
\[ \therefore 2x = k\pi \text{ for any integer } k \]
\[ \therefore x = \frac{k\pi}{2} \]

On the domain \( 0 \leq x \leq 2\pi \), \( f'(x) = 0 \)
when \( x = 0, \frac{\pi}{2}, \pi, 2\frac{\pi}{2}, \text{ and } 2\pi \).
Sign diagram for \( f'(x) \) is:

There are local maxima at \((0, 1), (\pi, 1), (2\pi, 1)\) and local minima at \(\left( \frac{\pi}{2}, -1 \right), \left( 2\frac{\pi}{2}, -1 \right)\).

4  If \( f(x) = \sin^2 x \) then \( f'(x) = 2 \sin x \cos x = \sin (2x) \)
\[ \therefore f'(x) = 0 \text{ when } \sin (2x) = 0 \]

Using 8, we know that on the domain \( 0 \leq x \leq 2\pi \)
\[ f'(x) = 0 \text{ when } x = 0, \frac{\pi}{2}, \pi, 2\frac{\pi}{2}, \text{ and } 2\pi \).
Sign diagram for \( f'(x) \) is:

There are local minima at \((0, 0), (\pi, 0), (2\pi, 0)\) and local maxima at \(\left( \frac{\pi}{2}, 1 \right), \left( 2\frac{\pi}{2}, 1 \right)\).
If \( f(x) = e^{\sin x} \) then \( f'(x) = e^{\sin x} \times \cos x \)

\[
\therefore f'(x) = 0 \text{ when } \cos x e^{\sin x} = 0
\]

\[
\therefore \cos x = 0 \quad \{e^{\sin x} > 0 \text{ for all } x\}
\]

\[
\therefore x = \frac{\pi}{2} + k\pi, \quad k \text{ an integer}
\]

On the domain \( 0 \leq x \leq 2\pi \), \( f'(x) = 0 \)

when \( x = \frac{\pi}{2}, \frac{3\pi}{2} \).

Sign diagram for \( f'(x) \):

There is a local maximum at \( (\frac{\pi}{2}, e) \)

and a local minimum at \( (\frac{3\pi}{2}, \frac{1}{e}) \).

If \( f(x) = \sin(2x) + 2 \cos x \) then \( f'(x) = 2 \cos(2x) - 2 \sin x \)

\[
\therefore f'(x) = 0 \text{ when } 2 \cos(2x) - 2 \sin x = 0
\]

\[
\therefore 2(1 - 2 \sin^2 x) - 2 \sin x = 0
\]

\[
\therefore -2(2 \sin^2 x + \sin x - 1) = 0
\]

\[
\therefore -2(2 \sin x - 1)(\sin x + 1) = 0
\]

\[
\therefore \text{when } \sin x = \frac{1}{2} \text{ or } \sin x = -1
\]

On the domain \( 0 \leq x \leq 2\pi \), when \( x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \).

Sign diagram of \( f'(x) \):

\[
f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) + 2 \cos\left(\frac{\pi}{6}\right)
\]

\[
= \frac{\sqrt{3}}{2} + 2 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}
\]

\[
f\left(\frac{5\pi}{6}\right) = \sin\left(\frac{10\pi}{6}\right) + 2 \cos\left(\frac{5\pi}{6}\right)
\]

\[
= -\frac{\sqrt{3}}{2} + 2(-\frac{\sqrt{3}}{2}) = -\frac{3\sqrt{3}}{2}
\]

\[
f\left(\frac{3\pi}{2}\right) = \sin(3\pi) + 2 \cos\left(\frac{3\pi}{2}\right)
\]

\[
= 0 + 2 \times 0 = 0
\]

\[
\therefore \text{there is a local maximum at } \left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right),
\]

a local minimum at \( \left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right) \)

and a stationary point of inflection at \( \left(\frac{3\pi}{2}, 0\right) \).

Let the cubic polynomial be

\[
P(x) = ax^3 + bx^2 + cx + d
\]

\[
\therefore P'(x) = 3ax^2 + 2bx + c \quad \ldots (1)
\]

Now \( (0, 2) \) lies on \( P(x) \), so \( P(0) = 2 \)

\[
\therefore a(0) + b(0) + c(0) + d = 2
\]

\[
\therefore d = 2
\]

The tangent at \( (0, 2) \) is \( y = 9x + 2 \), so

\[
P'(0) = 9
\]

\[
\therefore 3a(0) + 2b(0) + c = 9
\]

\[
\therefore c = 9 \quad \ldots (2)
\]

There is a stationary point at \( (-1, -7) \), so

\[
P'(-1) = 0
\]

\[
\therefore 3a(-1)^2 + 2b(-1) + c = 0 \quad \{\text{using (1)}\}
\]

\[
\therefore 3a - 2b + c = 0
\]

So, using (2), \( 3a - 2b = -9 \quad \ldots (3) \)

Finally, \( (-1, -7) \) lies on \( P(x) \)

\[
\therefore a(-1)^3 + b(-1)^2 + c(-1) + d = -7
\]

\[
\therefore -a + b - 9 + 2 = -7
\]

\[
\therefore b - a = 0
\]

\[
\therefore a = b
\]

So, using (3), \( 3a - 2a = -9 \)

\[
\therefore a = -9
\]

\[
\therefore a = b = -9
\]

\[
\therefore P(x) = -9x^3 - 9x^2 + 9x + 2
\]
10 \[ f(x) = x^3 - 12x - 2, \] for \(-3 \leq x \leq 5\)

\[
f'(x) = 3x^2 - 12 = 3(x + 2)(x - 2)
\]

which is 0 when \(x = -2\) or 2

\[
\therefore \text{the greatest value is 63 when } x = 5, \text{ and the least value is -18 when } x = 2.
\]

\[ f(x) = 4 - 3x^2 + x^3, \] for \(-2 \leq x \leq 3\)

\[
f'(x) = -6x + 3x^2 = 3x(x - 2)
\]

which is 0 when \(x = 0\) or 2

\[
\therefore \text{greatest value is 4 when } x = 0 \text{ or } x = 3, \text{ least value is -16 when } x = -2.
\]

11 \[ y = 4e^{-x} \sin x \]

\[ \frac{dy}{dx} = -4e^{-x} \sin x + 4e^{-x} \cos x \]

\[ \therefore \text{stationary points occur when } -4e^{-x} \sin x + 4e^{-x} \cos x = 0 \]

\[ 4e^{-x}(\cos x - \sin x) = 0 \]

\[ \therefore \cos x - \sin x = 0 \quad \{e^{-x} > 0 \text{ for all } x\} \]

\[ \therefore \sin x = \cos x \]

\[ \therefore \tan x = 1 \]

\[ \therefore x = \frac{\pi}{4} + k\pi, \quad k \text{ an integer} \]

Sign diagram of \(\frac{dy}{dx}\) is:

\[ \cdots \quad - \quad \frac{\pi}{2} \quad \frac{\pi}{4} \quad \frac{3\pi}{4} \quad + \quad \cdots \quad x \]

\[ \therefore y = 4e^{-x} \sin x \] has a local maximum when \(x = \frac{\pi}{4} \).

12 Consider \[ f(x) = \frac{\ln x}{x} \]

\[ f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} = \frac{1 - \ln x}{x^2} \]

\[ f'(x) = 0 \quad \text{when} \quad 1 - \ln x = 0 \]

\[ \therefore \ln x = 1 \]

\[ \therefore x = e \]

Sign diagram of \(f'(x)\) is:

\[ 0 \quad + \quad \frac{1}{e} \quad - \quad x \]

Now \(f(e) = \frac{\ln e}{e} = \frac{1}{e}\)

\[ \therefore \text{there is a local maximum at } (e, \frac{1}{e}) \]

\[ f(x) \leq \frac{1}{e} \text{ for all } x, \text{ and so } \frac{\ln x}{x} \leq \frac{1}{e} \text{ for all } x > 0 \]

13 \[ f(x) = x - \ln x \]

\[ f'(x) = 1 - \frac{1}{x} = \frac{x - 1}{x} \]

and the sign diagram of \(f'(x)\) is:

\[ 0 \quad - \quad 1 \quad + \quad x \]

\[ \therefore f(x) \] has a local minimum at \((1, 1 - \ln 1)\) or \((1, 1)\). This is the only turning point.

\[ f(x) \geq 1 \text{ for all } x > 0 \]

\[ \therefore x - \ln x \geq 1 \]

\[ \therefore \ln x \leq x - 1 \text{ for all } x > 0 \]
EXERCISE 16D.1

1. a) The turning points of \( y = f(x) \) are point B, a local minimum, and point D, a local maximum.

b) The inflection point of \( y = f(x) \) is point C, a non-stationary point of inflection.

2. a) \( f(x) = x^3 + 3 \)
   \[ f'(x) = 2x \]
   \[ f''(x) = 2 \]
   Since \( f''(x) \neq 0 \), no points of inflection exist.

b) \( f(x) = 2 - x^3 \)
   \[ f'(x) = -3x^2 \]
   \[ f''(x) = -6x \]
   Now \( f''(x) = 0 \) when \( x = 0 \), and \( f'(0) = 0 \).
   \[ \therefore \text{there is a stationary inflection at (0, 2).} \]

c) \( f(x) = x^3 - 6x^2 + 9x + 1 \)
   \[ f'(x) = 3x^2 - 12x + 9 \]
   \[ = 3(x^2 - 4x + 3) \]
   \[ = 3(x - 3)(x - 1) \]
   and \( f''(x) = 6x - 12 = 6(x - 2) \)
   Now \( f''(x) = 0 \) when \( x = 2 \) and \( f'(2) \neq 0 \).
   \[ \therefore \text{there is a non-stationary inflection at (2, f(2)) which is (2, 3).} \]

d) \( f(x) = x^3 + 6x^2 + 12x + 5 \)
   \[ f'(x) = 3x^2 + 12x + 12 \]
   \[ = 3(x^2 + 4x + 4) \]
   \[ = 3(x + 2)^2 \]
   and \( f''(x) = 6x + 12 = 6(x + 2) \)
   Now \( f''(x) = 0 \) when \( x = -2 \) and \( f'(-2) = 0 \).
   \[ \therefore \text{there is a stationary inflection at } (-2, f(-2)) \text{ which is } (-2, -3). \]

e) \( f(x) = -3x^4 - 8x^3 + 2 \)
   \[ f'(x) = -12x^3 - 24x^2 \]
   \[ = -12x^2(x + 2) \]
   and \( f''(x) = -36x^2 - 48x \)
   \[ = -12x(3x + 4) \]
   \[ \therefore \text{there is a stationary inflection at (0, 2), and a non-stationary inflection at } \]
   \[ (-\frac{2}{3}, f(-\frac{2}{3})), \text{ which is } (-\frac{2}{3}, \frac{310}{27}). \]

f) \( f(x) = 3 - \frac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}} \)
   \[ f'(x) = \frac{1}{2}x^{-\frac{3}{2}} \]
   and \( f''(x) = -\frac{3}{4}x^{-\frac{5}{2}} = \frac{-3}{4x^2\sqrt{x}} \)
   Now \( f''(x) \neq 0 \) for all \( x \).
   \[ \therefore \text{there are no points of inflection.} \]
3. \( f(x) = x^2 \)
   \[ f'(x) = 2x \text{ which has sign diagram:} \]
   \[ f''(x) = 2 \]
   I. There is a local minimum at (0, 0).
   II. There are no points of inflection as \( f''(x) \neq 0 \).
   III. \( f(x) \) is increasing when \( x \geq 0 \), and decreasing when \( x \leq 0 \).
   IV. \( f(x) \) is concave up for all \( x \) as \( f''(x) > 0 \) for all \( x \).

b. \( f(x) = x^3 \)
   \[ f'(x) = 3x^2 \text{ which has sign diagram:} \]
   and \( f''(x) = 6x \text{ which has sign diagram:} \]
   I. A stationary inflection at (0, 0).
   II. A stationary inflection at (0, 0).
   III. \( f(x) \) is increasing for all real \( x \).
   IV. \( f(x) \) is concave up when \( x \geq 0 \), and concave down when \( x \leq 0 \).

c. \( f(x) = \sqrt{x} \)
   \[ f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \text{ which has sign diagram:} \]
   and \( f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}} \text{ which has sign diagram:} \]
   I. There are no stationary points as \( f'(x) \neq 0 \).
   II. There are no points of inflection as \( f''(x) \neq 0 \).
   III. \( f(x) \) is increasing for all \( x \geq 0 \), never decreasing.
   IV. \( f(x) \) is concave down for all \( x \geq 0 \) as \( f''(x) < 0 \)
      for all \( x > 0 \), never concave up.

d. \( f(x) = x^3 - 3x^2 - 24x + 1 \)
   \[ f'(x) = 3x^2 - 6x - 24 \]
   \[ = 3(x^2 - 2x - 8) \]
   \[ = 3(x - 4)(x + 2) \text{ which has sign diagram:} \]
   and \( f''(x) = 6x - 6 \text{ which has sign diagram:} \]
   I. \( f(-2) = 29 \), \( f(4) = -79 \), so there is a local maximum
      at \((-2, 29)\), and a local minimum at \((4, -79)\).
   II. \( f(1) = -25 \), so there is a non-stationary inflection
      at \((1, -25)\).
   III. \( f(x) \) is increasing for \( x \leq -2 \) and \( x \geq 4 \),
      and decreasing for \(-2 \leq x \leq 4 \).
   IV. \( f(x) \) is concave down for \( x \leq 1 \), and
      concave up for \( x \geq 1 \).
\( f(x) = 3x^4 + 4x^3 - 2 \)
\[ f'(x) = 12x^3 + 12x^2 \]
\[ = 12x^2(x + 1) \quad \text{which has sign diagram:} \]
and 
\[ f''(x) = 36x^2 + 24x \]
\[ = 12x(3x + 2) \quad \text{which has sign diagram:} \]

I. There is a local minimum at \((-1, f(-1))\) which is \((-1, -3)\), and a stationary inflection at \((0, -2)\).

II. There is a non-stationary inflection at \((-\frac{2}{3}, f(-\frac{2}{3}))\)
which is \((-\frac{2}{3}, -\frac{20}{9})\), and a stationary inflection at \((0, -2)\).

III. \( f(x) \) is increasing for \( x \geq -1 \), and decreasing for \( x \leq -1 \).

IV. \( f(x) \) is concave down for \(-\frac{2}{3} \leq x \leq 0\), and concave up for \( x \leq -\frac{2}{3} \) and \( x \geq 0 \).

\[ f(x) = (x - 1)^4 \]
\[ f'(x) = 4(x - 1)^3 \quad \text{which has sign diagram:} \]
and 
\[ f''(x) = 12(x - 1)^2 \quad \text{which has sign diagram:} \]

I. There is a local minimum at \((1, 0)\).

II. Since there is no sign change in \( f''(x) \) at \( x = 1 \),
there are no points of inflection.

III. \( f(x) \) is increasing for \( x \geq 1 \), and decreasing for \( x \leq 1 \).

IV. \( f(x) \) is concave up for all \( x \).

\[ f(x) = x^4 - 4x^3 + 3 \]
\[ f'(x) = 4x^3 - 8x = 4x(x^2 - 2) \]
\[ = 4x(x + \sqrt{2})(x - \sqrt{2}) \quad \text{which has sign diagram:} \]
\[ f''(x) = 12x^2 - 8 = 4(3x^2 - 2) \]
\[ = 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2}) \quad \text{which has sign diagram:} \]

I. There is a local maximum at \((0, 3)\), and
\( f(-\sqrt{2}) = f(\sqrt{2}) = -1 \), so there are
local minima at \((\sqrt{2}, -1)\) and \((-\sqrt{2}, -1)\).

II. \( f\left(\frac{\sqrt{2}}{3}\right) = f\left(-\frac{\sqrt{2}}{3}\right) = \frac{7}{9} \), so there are
non-stationary inflections at
\(\left(\frac{\sqrt{2}}{3}, \frac{7}{9}\right)\) and \((-\frac{\sqrt{2}}{3}, \frac{7}{9})\).

III. \( f(x) \) is increasing for \(-\sqrt{2} \leq x \leq 0\) and
\( x \geq \sqrt{2} \), and decreasing for \( x \leq -\sqrt{2} \)
and \( 0 \leq x \leq \sqrt{2} \).

IV. \( f(x) \) is concave down for \(-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}} \),
and concave up for \( x \leq -\sqrt{\frac{2}{3}} \) and \( x \geq \sqrt{\frac{2}{3}} \).
\[ f(x) = 3 - \frac{4}{\sqrt{x}} = 3 - 4x^{-\frac{1}{2}}, \quad x > 0 \]

\[ f'(x) = 2x^{-\frac{3}{2}} = \frac{2}{x^{\frac{3}{2}}} \quad \text{with sign diagram:} \quad + \quad \text{at} \quad 0 \quad \rightarrow \quad x \]

and \[ f''(x) = -3x^{-\frac{5}{2}} = -\frac{3}{x^{\frac{5}{2}}} \quad \text{with sign diagram:} \quad - \quad \text{at} \quad 0 \quad \rightarrow \quad x \]

\[ \text{I} \quad \text{There are no stationary points as } f'(x) \neq 0. \\
\text{II} \quad \text{There are no points of inflection as } f''(x) \neq 0. \\
\text{III} \quad f(x) \text{ is increasing for all } x > 0 \text{ and never decreasing.} \\
\text{IV} \quad f(x) \text{ is concave down for all } x > 0 \text{ and never concave up.} \]

4. a. Consider \[ f(x) = e^{2x} - 3 \]

\[ f(x) \text{ cuts the } x\text{-axis at } A \text{ when } f(x) = 0 \]
\[ e^{2x} - 3 = 0 \]
\[ e^{2x} = 3 \]
\[ 2x = \ln 3 \]
\[ x = \frac{\ln 3}{2} \]
\[ \therefore \text{ A is } \left( \frac{\ln 3}{2}, 0 \right) \]

b. \[ f(x) = e^{2x} - 3 \]
\[ f'(x) = 2e^{2x} \]

Since \[ e^{2x} > 0 \text{ for all } x, \quad f'(x) > 0 \text{ for all } x, \text{ and hence } f(x) \text{ is increasing for all } x. \]

c. \[ f''(x) = 4e^{2x}, \text{ which is always } > 0. \]
\[ \therefore \quad f(x) \text{ is concave up for all } x. \]

d. \[ y = e^{2x} - 3 \]

\[ x = -\infty, \quad e^{2x} \to 0, \quad \text{so } \quad e^{2x} - 3 \to -3^+ \]
\[ \therefore \quad y = -3 \quad \text{is a horizontal asymptote.} \]

5. a. The \( x \)-intercepts occur when \( y = 0 \)

For \( f(x) = e^x - 3, \quad e^x - 3 = 0 \) and for \( g(x) = 3 - \frac{5}{e^x}, \quad 3 - \frac{5}{e^x} = 0 \)
\[ e^x = 3 \]
\[ x = \ln 3 \]
\[ \therefore \quad 3e^x - 5 = 0 \]
\[ \therefore \quad e^x = \frac{5}{3} \]
\[ x = \ln \left( \frac{5}{3} \right) \]

\[ \therefore \quad f(x) \text{ has } x\text{-intercept } \ln 3 \]

and \[ g(x) \text{ has } x\text{-intercept } \ln \left( \frac{5}{3} \right) \]

The \( y \)-intercepts occur when \( x = 0 \)

Now \( f(0) = e^0 - 3 = -2 \) and \( g(0) = 3 - \frac{5}{e^0} = 3 - 5 = -2 \)
\[ \therefore \quad \text{both } f(x) \text{ and } g(x) \text{ have } y\text{-intercept } -2. \]
\( f(x) \) and \( g(x) \) meet when
\[
\begin{align*}
e^x - 3 &= 3 - 5e^{-x} \\
e^{2x} - 3e^x &= 3e^x - 5 \quad \{ \times e^x \} \\
e^{2x} - 6e^x + 5 &= 0 \\
(e^x - 5)(e^x - 1) &= 0 \\
e^x &= 5 \text{ or } 1 \\
x &= \ln 5 \text{ or } 0
\end{align*}
\]
Now \( f(\ln 5) = e^{\ln 5} - 3 = 5 - 3 = 2 \)
and \( f(0) = -2 \)
\( \therefore f(x) \) and \( g(x) \) meet at \((\ln 5, 2)\) and \((0, -2)\).

\( \frac{dy}{dx} = e^x + 3e^{-x} = e^x + \frac{3}{e^x} \)
Since \( e^x > 0 \) for all \( x \),
\( \frac{dy}{dx} > 0 \) for all \( x \)
\( \therefore \) the function is increasing for all \( x \)

\( f(x) = \ln(2x - 1) - 3 \)
\( \begin{align*}
f(x) &= 0 \quad \text{when} \quad \ln(2x - 1) = 3 \\
\therefore 2x - 1 &= e^3 \\
\therefore 2x &= e^3 + 1 \\
\therefore x &= \frac{e^3 + 1}{2} \approx 10.5 \\
\therefore \text{the x-intercept is} \quad \frac{e^3 + 1}{2}
\end{align*} \)
\( f(0) \) cannot be found as \( \ln(-1) \) is not defined. \( \therefore \) there is no \( y \)-intercept.

\( f'(x) = \frac{2}{2x - 1} \quad \therefore f'(1) = \frac{2}{2 - 1} = 2 \quad \therefore \text{gradient of tangent} = 2 \)
\( \ln(2x - 1) \) has meaning provided \( 2x - 1 > 0 \)
\( \therefore 2x > 1 \) and so \( x > \frac{1}{2} \)
\( \therefore \text{the domain of} \ f \ \text{is} \ \{ x \mid x > \frac{1}{2} \} \)
\[ f'(x) = 2(2x - 1)^{-1} \]
\[ f''(x) = -2(2x - 1)^{-2}(2) \]
\[ f'(x) = \frac{-4}{(2x - 1)^2}, \quad x > \frac{1}{2} \]

\[ \therefore \text{ provided } x > \frac{1}{2}, \quad f''(x) < 0 \]
\[ \therefore f(x) \text{ is concave down for all } x \text{ in the domain of } f. \]

\[ f(x) = \ln x \text{ is defined for all } x > 0. \]

\[ f'(x) = \frac{1}{x} \text{ which is } > 0 \text{ for all } x > 0 \]
\[ \therefore f(x) \text{ is increasing on } x > 0; \quad \text{its gradient is always positive.} \]
\[ f''(x) = -x^{-2} = \frac{-1}{x^2} \text{ which is } < 0 \text{ for all } x > 0 \]
\[ \therefore f(x) \text{ is concave down on } x > 0. \]

\[ f(x) = \ln x \]
\[ y = 1, \quad 1 = \ln x \]
\[ \therefore x = e^1 = e \]
\[ \therefore \text{ the point of contact is } (e, 1) \]

\[ \text{Now } \frac{dy}{dx} = \frac{1}{x} \]
\[ \therefore \text{ at } (e, 1), \quad \frac{dy}{dx} = \frac{1}{e} \]

\[ \therefore \text{ the gradient of the tangent is } \frac{1}{e}, \text{ and the gradient of the normal is } -e \]

\[ \therefore \text{ the equation of the normal is } \frac{y - 1}{x - e} = -e \]
\[ \therefore y - 1 = -e(x - e) \]
\[ \therefore y = -ex + e^2 \]
\[ \therefore y = -ex + 1 + e^2 \]

Consider \[ f(x) = \frac{e^x}{x} \]

\[ e^x \neq 0 \text{ for all } x, \text{ so } f(x) \neq 0 \text{ and there is no } x\text{-intercept.} \]

\[ f(0) = \frac{e^0}{0} \text{ is undefined, so there is also no } y\text{-intercept.} \]

\[ \text{As } x \to +\infty \quad f(x) \to \infty, \quad \text{and as } x \to -\infty, \quad f(x) \to 0^- \]
\[ \text{(As } x \to 0^+, \quad y \to +\infty, \quad \text{and as } x \to 0^-, \quad y \to -\infty \text{)} \]
\[ \therefore x = 0 \text{ is a vertical asymptote.} \]

\[ \text{Using the quotient rule, } \quad f'(x) = \frac{e^x(x - 1) - e^x x}{x^2} \]
\[ \quad = \frac{e^x(x^2 - 2x + 2)}{x^3} \]

\[ f(1) = \frac{e^1}{1} = e, \quad \text{so there is a local minimum at } (1, e). \]

\[ \text{Using the product and quotient rules,} \]
\[ f''(x) = \frac{[e^x(x - 1) + e^x]x^2 - e^x(x - 1)2x}{x^4} \]
\[ = \frac{e^x(x^2 - 2x + 2)}{x^3} \]

\[ \text{with sign diagram:} \]

\[ f(x) \text{ is concave up for } x > 0. \]
\[ f(x) \text{ is concave down for } x < 0. \]
Now \( f'(x) = \frac{e^x(x-1)}{x^2} \)

\[ f'(-1) = \frac{e^{-1}(-1-1)}{(-1)^2} = -2 \frac{2}{e} \]

\[ \therefore \text{the gradient of the tangent is } -\frac{2}{e} \]

When \( x = -1, y = \frac{e^{-1}}{-1} = -\frac{1}{e} \)

\[ \therefore \text{the equation of tangent is } \frac{y - \frac{1}{e}}{x - (-1)} = -\frac{2}{e} \]

\[ \therefore \frac{y + \frac{1}{e}}{x + 1} = -\frac{2}{e} \]

\[ \therefore e(y + \frac{1}{e}) = -2(x + 1) \]

\[ \therefore ey + 1 = -2x - 2 \]

\[ \therefore ey = -2x - 3 \]

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \]

\[ f'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \]

\[ \therefore f'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (-x) \]

\[ \therefore f''(x) = \frac{1}{\sqrt{2\pi}} \left( (-1) e^{-\frac{1}{2}x^2} + (-x) e^{-\frac{1}{2}x^2} (-x) \right) \]  

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (x^2 - 1) \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (x + 1)(x - 1) \]

\[ \therefore \text{which has sign diagram: } \begin{array}{ccc} + & - & - \\ -1 & & 1 \\ & + & + \end{array} \]

\[ \text{Now } f(1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sqrt{2e\pi}} \text{ and } f(-1) = \frac{-1}{\sqrt{2e\pi}} \]

\[ \therefore \text{there are non-stationary points of inflection at } \left(1, \frac{1}{\sqrt{2e\pi}} \right) \text{ and } \left(-1, \frac{1}{\sqrt{2e\pi}} \right) . \]

As \( x \to \infty, e^{-\frac{1}{2}x^2} \to 0^+ \)

\[ \therefore f(x) \to 0^+ \]

As \( x \to -\infty, e^{-\frac{1}{2}x^2} \to 0^+ \)

\[ \therefore f(x) \to 0^+ \]

**EXERCISE 16D.2**

1. \( f(x) \) is quadratic, so \( f'(x) \) will be linear and \( f''(x) \) will be constant.

\( f(x) \) is decreasing for \( x < 1 \) and increasing for \( x > 1 \)

\[ \therefore f'(x) \leq 0 \text{ for } x < 1 \text{ and } f'(x) \geq 0 \text{ for } x > 1 \]

\[ \therefore f'(x) \text{ is an increasing linear function which cuts the } x\text{-axis at } 1. \]

As \( f'(x) \) is increasing, \( f''(x) > 0. \)
\( f(x) \) is cubic, so \( f'(x) \) will be quadratic and \( f''(x) \) will be linear.

- \( f(x) \) has turning points at \( x \approx \pm 1 \)
  \[ f' \] cuts the \( x \)-axis at these points.
- \( f(x) \) has a non-stationary inflection point at \( x = 0 \)
  \[ f' \] has a turning point at \( x = 0 \), and \( f''(0) = 0 \).
- \( f(x) \) is concave down for \( x \leq 0 \) and concave up for \( x \geq 0 \)
  \[ f' \] is decreasing for \( x \leq 0 \) and increasing for \( x \geq 0 \)
  and \( f''(x) \leq 0 \) for \( x \leq 0 \) and \( \geq 0 \) for \( x \geq 0 \).

\( f(x) \) is cubic, so \( f'(x) \) will be quadratic and \( f''(x) \) will be linear.

- \( f(x) \) has turning points at \( x \approx 1 \) and \( x = 3 \)
  \[ f' \] cuts the \( x \)-axis at these points.
- \( f(x) \) has a non-stationary inflection point at \( x = 2 \)
  \[ f' \] has a turning point at \( x \approx 2 \), and \( f''(2) = 0 \)
- \( f(x) \) is concave down for \( x \leq 2 \) and concave up for \( x \geq 2 \)
  \[ f' \] is decreasing for \( x \leq 2 \) and increasing for \( x \geq 2 \)
  and \( f''(x) \leq 0 \) for \( x \leq 2 \) and \( \geq 0 \) for \( x \geq 2 \).

**2** a \( f'(x) \) has sign diagram:

- \( f(x) \) is increasing for \( x \leq -3 \), and decreasing for \( x \geq -3 \)
- \( f(x) \) has a local maximum at \( x = -3 \)
- \( f'(x) \) has a turning point at \( x \approx -1.7 \).
  At this point, \( f''(x) = 0 \), but \( f'(x) \neq 0 \)
- \( f(x) \) has a non-stationary inflection point here.
  \( f'(x) \) has another turning point at \( x = 1 \).
  At this point, \( f''(x) = 0 \) and \( f'(x) = 0 \)
- \( f(x) \) has a stationary inflection point at \( x = 1 \).
  A possible graph of \( f(x) \) is shown alongside:

\( f'(x) \) has sign diagram:

- \( f(x) \) has a local minimum at \( x = -2 \) and a local maximum at \( x = 4 \)
- \( f'(x) \) has a turning point at \( x \approx 1 \).
  At this point, \( f''(x) = 0 \), but \( f'(x) \neq 0 \)
- \( f(x) \) has a non-stationary inflection point at \( x \approx 1 \).
  A possible graph of \( f(x) \) is shown alongside:
**REVIEW SET 16A**

1. Consider \( y = -2x^2 \). When \( x = -1 \), \( y = -2(-1)^2 = -2 \), so the point of contact is \((-1, -2)\).

   Now \( \frac{dy}{dx} = -4x \)

   \( \therefore \) the tangent has equation \( \frac{y - (-2)}{x - (-1)} = 4 \)

   \( \therefore \) at \( x = -1 \), \( \frac{dy}{dx} = -4(-1) = 4 \)

   or \( y = 4x + 2 \)

2. Consider \( y = \frac{1 - 2x}{x^2} \). When \( x = 1 \), \( y = \frac{1 - 2(1)}{1^2} = -1 \), so the point of contact is \((1, -1)\).

   Since \( \frac{1}{x^2} - \frac{2}{x} \cdot \frac{dy}{dx} = -2x^{-3} + 2x^{-2} = -\frac{2}{x^3} + \frac{2}{x^2} \)

   \( \therefore \) at \( x = 1 \), \( \frac{dy}{dx} = -2 + 2 = 0 \)

   So, the tangent is a horizontal line, and the normal must be a vertical line of the form \( x = k \).

   As the normal passes through \((1, -1)\), its equation must be \( x = 1 \).

3. (a) The vertical asymptote is \( x + 3 = 0 \) or \( x = -3 \).

   (b) When \( y = 0 \), \( \frac{3x - 2}{x + 3} = 0 \)

   \( \therefore 3x - 2 = 0 \)

   \( \therefore x = \frac{2}{3} \)

   \( \therefore \) the \( x \)-intercept is \( \frac{2}{3} \).

   When \( x = 0 \), \( f(0) = -\frac{3}{2} \)

   \( \therefore \) the \( y \)-intercept is \( -\frac{3}{2} \).

4. \( y = e^{-x^2} \) so when \( x = 1 \), \( y = e^{-1} = \frac{1}{e} \)

   \( \therefore \) the point of contact is \((1, \frac{1}{e})\)

   Now \( \frac{dy}{dx} = -2xe^{-x^2} \)

   \( \therefore \) when \( x = 1 \), \( \frac{dy}{dx} = -2e^{-1} \)

   \( \therefore \) the gradient of the tangent is \( -\frac{2}{e} \)

   and the gradient of the normal is \( \frac{e}{2} \)

   \( \therefore \) the equation of the normal is \( \frac{y - \frac{1}{e}}{x - 1} = \frac{e}{2} \)

   \( \therefore \) \( 2 \left( y - \frac{1}{e} \right) = e(x - 1) \)

   \( \therefore \) \( 2y - \frac{2}{e} = ex - e \)

   \( \therefore \) \( 2y = ex + \frac{2}{e} - e \)

   \( \therefore \) \( y = \frac{e}{2}x + \frac{1}{e} - \frac{e}{2} \)

5. \( y = x \tan x \)

   \( \therefore \) \( \frac{dy}{dx} = 1 \times \tan x + x \times \left( \frac{1}{\cos^2 x} \right) \)

   \( = \tan x + \frac{x}{\cos^2 x} \)

   Now \( \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \) and \( \tan \frac{\pi}{4} = 1 \)

   \( \therefore \) at \( x = \frac{\pi}{4} \), \( y = \frac{\pi}{4} \),

   and \( \frac{dy}{dx} = 1 + \frac{\frac{\pi}{4}}{(1/\sqrt{2})^2} = 1 + \frac{\pi}{2} \)

   \( \therefore \) the equation of the tangent is \( \frac{y - \frac{\pi}{4}}{x - \frac{\pi}{4}} = 1 + \frac{\pi}{2} \)

   \( \therefore \) \( y - \frac{\pi}{4} = (1 + \frac{\pi}{2})(x - \frac{\pi}{4}) \)

   \( = x - \frac{\pi}{4} + \frac{\pi}{2}x - \frac{\pi^2}{8} \)

   \( \therefore \) \( y = (1 + \frac{\pi}{2})x - \frac{\pi^2}{8} \)

   \( \therefore \) \( 2y = (2 + \pi)x - \frac{\pi^2}{4} \)

   \( \therefore \) \((2 + \pi)x - 2y = \frac{\pi^2}{4} \) as required
6 \[ y = \frac{ax + b}{\sqrt{x}} = a\sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}} \]

\[ \therefore \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}} = \frac{a}{2\sqrt{x}} - \frac{b}{2x\sqrt{x}} \]

The equation of the tangent at \( x = 1 \)

\[ \text{is } 2x - y = 1 \]

or \( y = 2x - 1 \)

so the gradient of the tangent is 2

\[ \therefore \text{at } x = 1, \frac{dy}{dx} = \frac{a}{2} - \frac{b}{2} = 2 \]

\[ \therefore a - b = 4 \]

\[ \therefore a = b + 4 \quad \ldots \quad (1) \]

Also at \( x = 1 \), the tangent touches the curve

\[ \frac{a(1) + b}{\sqrt{1}} = 2(1) - 1 \]

\[ \therefore a + b = 1 \]

\[ \therefore b + 4 + b = 1 \quad \text{\{using \( (1) \)} \]

\[ \therefore 2b = -3 \]

\[ \therefore b = -\frac{3}{2} \quad \text{and} \quad a = 4 - \frac{3}{2} = \frac{5}{2} \]

7 \[ f(x) = 4\ln(2x), \quad P(1, 4\ln 2) \]

\[ \therefore f'(x) = 4 \times \frac{2}{2x} = \frac{4}{x} \]

\[ \therefore \text{at } x = 1, \quad f'(1) = \frac{4}{1} = 4 \]

\[ \therefore \text{the tangent has equation} \]

\[ \frac{y - 4\ln 2}{x - 1} = 4 \]

\[ \therefore y - 4\ln 2 = 4x - 4 \]

\[ \therefore y = 4x + 4\ln 2 - 4 \]

8 a \[ f(x) = \frac{e^x}{x - 1} \]

Now \( f(0) = \frac{e^0}{-1} = -1 \) so the \( y \)-intercept is \(-1\).

b \( f(x) \) is defined for all \( x \neq 1 \).

c \[ f'(x) = \frac{e^x(x - 1) - e^x(1)}{(x - 1)^2} \quad \text{\{quotient rule\}} \]

\[ = \frac{e^x(x - 2)}{(x - 1)^2} \quad \text{and has sign diagram:} \]

\[ - \quad \frac{1}{1} \quad - \frac{1}{2} \quad + \quad x \]

\[ \therefore f'(x) \leq 0 \quad \text{for } x < 1 \quad \text{and} \quad 1 < x \leq 2 \quad \text{and} \quad f'(x) \geq 0 \quad \text{for } x \geq 2 \]

\[ \therefore f(x) \text{ is decreasing for } x < 1 \quad \text{and} \quad 1 < x \leq 2, \quad \text{and increasing for } x \geq 2. \]

\[ f''(x) = \frac{[e^x(x - 2) + e^x(1)](x - 1)^2 - e^x(x - 2)[2(x - 1)^4(1)]}{(x - 1)^4} \quad \text{\{product and quotient rules\}} \]

\[ = \frac{e^x(x - 2) + e^x(x - 1)^2 - 2e^x(x - 2)(x - 1)}{(x - 1)^4} \]

\[ = \frac{e^x(x - 1)(x - 1)^2 - 2e^x(x - 2)(x - 1)}{(x - 1)^4} \]

\[ = \frac{e^x(x - 1)[(x - 1)^2 - 2(x - 2)]}{(x - 1)^4} \]

\[ = \frac{e^x(x - 1)[x^2 - 2x + 1 - 2x + 4]}{(x - 1)^4} \]

\[ = \frac{e^x(x^2 - 4x + 5)}{(x - 1)^3} \quad \text{where the quadratic term has } \Delta < 0 \]

The sign diagram of \( f''(x) \) is: \[ - \quad \frac{1}{1} \quad + \quad x \]

\[ \therefore f''(x) > 0 \quad \text{for } x > 1 \]

and \( f''(x) < 0 \) for \( x < 1 \).

\[ \therefore f(x) \text{ is concave down for all } x < 1 \]

and concave up for all \( x > 1 \).
9. \( y = 2x^3 + ax + b \) \[ \frac{dy}{dx} = 6x^2 + a \]

Now as the gradient at \((-2, 33)\) is 10,

at \( x = -2 \), \[ \frac{dy}{dx} = 10 \]

\[ \therefore \ 10 = 6(-2)^2 + a \]

\[ \therefore \ a = -14 \]

\[ \therefore \ y = 2x^3 - 14x + b \]

10. \[ y = \frac{a}{(x + 2)^2} = a(x + 2)^{-2} \]

The gradient of the line (AB) is

\[ \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 4}{0 - 2} = \frac{4}{-2} = -2 \]

\[ \therefore \ \text{the equation of the tangent is} \]

\[ y - 8 = -2 \text{ or } y = -2x + 8 \]

Now \[ \frac{dy}{dx} = -2a(x + 2)^{-3} \]

so for the given tangent, \(-2a(x + 2)^{-3} = -2\)

\[ \therefore \ a = (x + 2)^3 \quad (1) \]

The line (AB) meets the curve where

\[ -2x + 8 = \frac{a}{(x + 2)^2} \]

\[ \therefore -2x + 8 = \frac{(x + 2)^3}{(x + 2)^2} \quad \{\text{using (1)}\} \]

\[ \therefore 2x + 8 = x + 2 \]

\[ \therefore 3x = -6 \]

\[ \therefore x = 2 \]

and so \( a = (2 + 2)^3 = 64 \)

11. \[ y = \frac{5}{\sqrt{x}} = 5x^{-\frac{1}{2}} \]

\[ \therefore \frac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} \]

\[ \therefore \ \text{the gradient of the tangent at the point} \]

\( (1, 5) \) is \(-\frac{5}{2}(1)^{-\frac{3}{2}} = -\frac{5}{2} \)

\[ \therefore \ \text{the equation of the tangent is} \]

\[ y - 5 = -\frac{5}{2}(x - 1) \]

\[ \therefore \ y = -\frac{5}{2}x + \frac{15}{2} \]

Then, since \((-2, 33)\) lies on the curve,

at \( x = -2 \), \[ y = 33 \]

\[ \therefore 2(-2)^3 - 14(-2) + b = 33 \]

\[ \therefore -16 + 28 + b = 33 \]

\[ \therefore b = 21 \]

12. At \( x = A \), \( f'(x) = 0 \) and \( f''(x) = 0 \)

\[ \therefore \ f(x) \ \text{has a stationary inflection point at } x = A. \]

At \( x = B \), \( f''(x) = 0 \) but \( f'(x) \neq 0 \)

\[ \therefore \ f(x) \ \text{has a non-stationary inflection point at } x = B. \]

\( f'(x) \) is above the \( x \)-axis for \( x \leq C \), and below the \( x \)-axis for \( x \geq C \)

\[ \therefore \ f(x) \ \text{is increasing for } x \leq C \ \text{and decreasing for } x \geq C, \ \text{so } f(x) \ \text{has a local maximum at } x = C. \]
13 \[ y = \ln(x^2 + 3) \quad \therefore \quad \frac{dy}{dx} = \frac{2x}{x^2 + 3} \]

When \( x = 0 \), \( \frac{dy}{dx} = 0 \) so the gradient of the tangent at this point is 0.

But when \( x = 0 \), \( y = \ln(0 + 3) = \ln 3 \)

\[ \therefore \text{the tangent is } y = \ln 3 \text{ which does not cut the x-axis.} \]

**REVIEW SET 16B**

1 \[ y = x^3 - 3x^2 - 9x + 2 \quad \therefore \quad \frac{dy}{dx} = 3x^2 - 6x - 9 \]

Horizontal tangents occur when \( \frac{dy}{dx} = 0 \)

\[ \therefore 3x^2 - 6x - 9 = 0 \]

\[ \therefore x^2 - 2x - 3 = 0 \]

\[ \therefore (x - 3)(x + 1) = 0 \]

\[ \therefore x = 3 \text{ or } x = -1 \]

When \( x = 3 \), the horizontal tangent has equation \( y = -25 \).

When \( x = -1 \), the horizontal tangent has equation \( y = 7 \).

2 Consider the tangent to \( y = x^2\sqrt{1-x} \) at \( x = -3 \).

When \( x = -3 \), \( y = (-3)^2\sqrt{1-(-3)} = 9\sqrt{4} = 18 \) \( \{y \geq 0\} \)

\[ \therefore \text{the point of contact is } (-3, 18). \]

Also, \( y = x^2\sqrt{1-x} \) is a product with \( u = x^2 \) and \( v = (1-x)^{\frac{1}{2}} \)

\[ \therefore u' = 2x \text{ and } v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) \]

\[ \therefore \frac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} - x^2(\frac{1}{2})(1-x)^{-\frac{1}{2}} \]

\[ \therefore \text{at } x = -3, \frac{dy}{dx} = 2(-3)(1-(-3))^{\frac{1}{2}} - (-3)^2(\frac{1}{2})(1-(-3))^{-\frac{1}{2}} \]

\[ = -6(2) - 9(\frac{1}{2})(\frac{1}{2}) \]

\[ = -\frac{57}{4} \]

\[ \therefore \text{the tangent at } (-3, 18) \text{ has equation } \frac{y-18}{x-(-3)} = -\frac{57}{4} \]

\[ \therefore 4y - 72 = -57x - 171 \]

\[ \therefore 4y = -57x - 99 \]

Now when \( x = 0 \), \( y = -\frac{99}{4} \) and when \( y = 0 \), \( x = -\frac{99}{57} \)

\[ \therefore \text{the area of } \triangle OAB = \frac{1}{2} \left( \frac{98}{4} \right) \left( \frac{99}{57} \right) = \frac{3247}{18} \approx 21.5 \text{ units}^2 \]

3 \[ a \quad f(x) = x^3 + ax, \quad a < 0 \]

\[ \therefore f'(x) = 3x^2 + a \]

\( f(x) \) has a turning point at \( x = \sqrt{2} \), so \( f'(\sqrt{2}) = 0 \)

\[ \therefore 3(\sqrt{2})^2 + a = 0 \]

\[ \therefore a = -6 \]

\[ b \quad f'(x) = 3x^2 - 6 = 3(x^2 - 2) = 3(x + \sqrt{2})(x - \sqrt{2}) \]

\[ \therefore f'(x) \text{ has sign diagram:} \]

\[ \therefore f(x) \text{ has a local maximum at } (-\sqrt{2}, (-\sqrt{2})^3 - 6(-\sqrt{2})) \text{ or } (-\sqrt{2}, 4\sqrt{2}) \]

and a local minimum at \((\sqrt{2}, (\sqrt{2})^3 - 6\sqrt{2})\) or \((\sqrt{2}, -4\sqrt{2})\).
4 \[ f(x) = e^{4x} + px + q \]
\[ \therefore f'(x) = 4e^{4x} + p \]
At the point where \( x = 0 \), the tangent to \( f(x) \) has equation \( y = 5x - 7 \), so \( f'(0) = 5 \)
\[ \therefore 4e^0 + p = 5 \]
\[ \therefore p = 1 \]

The tangent meets \( f(x) \) when \( x = 0 \) and \( y = 5(0) - 7 = -7 \), so \((0, -7)\) must lie on \( f(x) \) too.
\[ \therefore e^{4(0)} + p(0) + q = -7 \]
\[ \therefore 1 + q = -7 \]
\[ \therefore q = -8 \]

5 Consider the tangent to \( y = 2x^3 + 4x - 1 \) at \((1, 5)\).
\[ \frac{dy}{dx} = 6x^2 + 4 \quad \therefore \text{at} \quad x = 1, \quad \frac{dy}{dx} = 6(1)^2 + 4 = 10 \]
\[ \therefore \text{the tangent has equation} \quad \frac{y - 5}{x - 1} = 10 \quad \text{or} \quad y = 10x - 5 \]

Now the tangent meets the curve again where \( 10x - 5 = 2x^3 + 4x - 1 \)
\[ \therefore 2x^3 - 6x + 4 = 0 \]
\[ \therefore x^3 - 3x^2 + 2 = 0 \]
We know that \((x - 1)^3\) is a factor since the line is tangent to the curve at \( x = 1 \).
Consequently, \( x^3 - 3x^2 + 2 = (x - 1)^3(x + 2) = 0 \) \{since the constant term is 2\}
Thus \( x = -2 \) is the other solution and when \( x = -2 \), \( y = 2(-2)^3 + 4(-2) - 1 = -25 \)
\[ \therefore \text{the tangent meets the curve again at} \quad (-2, -25) \]

6 Consider \( y = 4(\alpha x + 1)^{-2} \).
When \( x = 0 \), \( y = 4(0 + 1)^{-2} = 4 \), so the point of contact is \((0, 4)\).
Now \[ \frac{dy}{dx} = -8(\alpha x + 1)^{-3}(\alpha) = \frac{-8\alpha}{(\alpha x + 1)^3} \quad \therefore \text{at} \quad x = 0, \quad \frac{dy}{dx} = -8\alpha \]
\[ \therefore \text{the tangent has equation} \quad \frac{y - 4}{x - 0} = -8\alpha \quad \text{or} \quad y - 4 = -8\alpha x \]
This tangent passes through \((1, 0)\), so \( 0 - 4 = -8\alpha \) \( \therefore \alpha = \frac{1}{2} \)

7 \[ f(x) = e^x - x \]
\[ a \quad f'(x) = e^x - 1 \]
so \( f'(x) = 0 \) when \( e^x = 1 \)
\[ \therefore x = 0 \]
\[ \therefore \quad \text{Sign diagram of} \quad f'(x) \quad \text{is:} \]

Now \( f(0) = e^0 - 0 = 1 \)
\[ \therefore \text{there is a local minimum at} \quad (0, 1) \]
\[ f'''(x) = e^x \]
\[ \therefore f'''(x) > 0 \text{ for all } x \]
\[ \therefore f(x) \text{ is concave up for all } x \]

Since a local minimum exists at \((0, 1)\),
\[ f(x) \geq 1 \text{ for all } x \]
\[ \therefore e^x - x \geq 1 \]
\[ \therefore e^x \geq x + 1 \text{ for all } x \]

8 Consider \[ y = \frac{x + 1}{x^2 - 2} \]
When \( x = 1 \), \[ y = \frac{1 + 1}{1^2 - 2} = -2 \]
\[ \therefore \text{ the point of contact is } (1, -2). \]
\[ y = \frac{x + 1}{x^2 - 2} \text{ is a quotient with } \]
\[ u = x + 1 \quad \text{and} \quad v = x^2 - 2 \]
\[ u' = 1 \quad \text{and} \quad v' = 2x \]
\[ \frac{dy}{dx} = \frac{(x^2 - 2) - (x + 1)(2x)}{(x^2 - 2)^2} \text{ quotient rule} \]
\[ \therefore \text{ at } x = 1, \quad \frac{dy}{dx} = \frac{(1 - 2) - 2(1 + 1)}{(1 - 2)^2} \]
\[ = \frac{-1 - 4}{1} = -5 \]
\[ \therefore \text{ the normal at } (1, -2) \text{ has gradient } \frac{1}{5}. \]
So the normal has equation \[ y - (-2) = \frac{1}{5}(x - 1) \]
\[ \therefore 5y + 10 = x - 1 \]
\[ \therefore y = \frac{1}{5}x - \frac{11}{5} \quad \text{ (or } x - 5y = 11) \]

10 Let \[ g(x) = ax^2 + bx + c. \]
\[ g(x) \text{ has } y\text{-intercept } (0, 3), \text{ so } g(0) = a(0)^2 + b(0) + c = 3 \]
\[ \therefore c = 3 \]
\[ \therefore g(x) = ax^2 + bx + 3 \]
The point \((2, 7)\) lies on \(g(x)\), so \[ g(2) = a(2)^2 + b(2) + 3 = 7 \]
\[ \therefore 4a + 2b = 4 \quad \ldots \quad (1) \]
Also, \[ g'(x) = 2ax + b \]
\[ \therefore g'(2) = 2a(2) + b = 4a + b \]
\[ \therefore \text{ the gradient of the tangent to } g(x) \text{ at } (2, 7) \text{ is } 4a + b. \]
But, the tangent at \((2, 7)\) passes through \((0, 11)\), so the gradient \[ \frac{7 - 11}{2 - 0} = -2 \]
\[ \therefore 4a + b = -2 \quad \ldots \quad (2) \]
Solving (1) and (2) simultaneously,
\[
\begin{align*}
4a + 2b &= 4 \\
4a + b &= -2
\end{align*}
\]
subtracting: \[ b = 6 \]
Using (2), \[ 4a + 6 = -2, \] so \[ a = -2 \]
So, \[ g(x) = -2x^2 + 6x + 3 \]
11  a  \[ f(x) = \sqrt{\cos x}, \quad 0 \leq x \leq 2\pi \]
\[ f(x) \text{ is defined when } \cos x \geq 0, \]
which is when \( 0 \leq x \leq \frac{\pi}{2} \)
and \( \frac{3\pi}{2} \leq x \leq 2\pi \).

b  \[ f(x) = (\cos x)^{\frac{1}{2}} \]
\[ \therefore f'(x) = \frac{1}{2} (\cos x)^{-\frac{1}{2}} (-\sin x) \]
\[ = \frac{-\sin x}{2\sqrt{\cos x}} \]
\[ \therefore f'(x) = 0 \quad \text{when } -\sin x = 0 \]
For \( 0 \leq x \leq 2\pi \), this is when \( x = 0, \pi, 2\pi \).
Sign diagram for \( f'(x) \) is:

\[ f(x) \text{ is increasing for } \frac{3\pi}{2} < x < 2\pi \]
and decreasing for \( 0 < x < \frac{\pi}{2} \).

12  \( f(x) \) has a turning point at \( x = 0 \)
\[ \therefore f'(0) = 0 \]
\( f(x) \) is increasing for \( x \geq 0 \),
except at the asymptote,
so \( f'(x) \) is positive for \( x \geq 0 \).
\( f(x) \) is decreasing for \( x \leq 0 \),
except at the asymptote,
so \( f'(x) \) is negative for \( x \leq 0 \).
As \( x \to \pm\infty \), \( f(x) \) becomes
closer to horizontal so \( f'(x) \to 0 \).

13  a  \[ y = \frac{4}{x} \]
\[ y = \frac{4}{x} \text{ cuts the } x\text{-axis when } y = 0 \]
\[ \therefore -\frac{4}{k^2} x + \frac{8}{k} = 0 \]
\[ \therefore \frac{4}{k^2} x = \frac{8}{k} \]
\[ \therefore x = 2k \]
\[ y = \frac{4}{k^2} x + \frac{8}{k} \text{ cuts the } y\text{-axis when } x = 0 \]
\[ \therefore y = \frac{8}{k} \]
\[ \therefore \text{A is at } (2k, 0) \]
\[ \therefore \text{B is at } (0, \frac{8}{k}) \]

b  For \( f(x) = \frac{4}{x} = 4x^{-1} \),
\[ f'(x) = -4x^{-2} = -\frac{4}{x^2} \quad \text{and } f'(k) = -\frac{4}{k^2}, \quad k > 0 \]
\[ \therefore \text{the gradient of the tangent to } f(x) \text{ at } (k, \frac{4}{k}) \text{ is } -\frac{4}{k^2} \]
\[ \therefore \text{the equation of the tangent is } \frac{y - \frac{4}{k}}{x - k} = -\frac{4}{k^2} \]
\[ \therefore ky^2 - 4k = -4x + 4k \]
\[ \therefore k^2 y = -4x + 8k \]
\[ \therefore y = \frac{4}{k^2} x + \frac{8}{k} \]

d  Area of triangle OAB = \( \frac{1}{2} (2k) \left( \frac{8}{k} \right) = 8 \text{ units}^2 \)
The gradient of the tangent to \( f(x) \) at \( \left( k, \frac{4}{k} \right) \) is \( -\frac{4}{k^2} \)

\[
\therefore \quad \text{the gradient of the normal to } f(x) \text{ at } \left( k, \frac{4}{k} \right) \text{ is } \frac{k^2}{4}
\]

\[
\therefore \quad \text{the equation of the normal is } \frac{y - \frac{4}{k}}{x - k} = \frac{k^2}{4}
\]

\[
\therefore \quad 4y - \frac{16}{k} = k^2x - k^3
\]

\[
\therefore \quad 4ky - k^3x = 16 - k^4
\]

This normal passes through \((1, 1)\), so \(4k - k^3 = 16 - k^4\)

\[
\therefore \quad k^4 - k^3 + 4k - 16 = 0
\]

\[
\therefore \quad (k - 2)(k + 2)(k^2 - k + 4) = 0 \quad \{\text{using technology}\}
\]

\[
\therefore \quad k = \pm 2
\]

But \( k > 0 \), so \( k = 2 \)

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**REVIEW SET 16C**

1. Consider the normal to the curve \( y = \frac{1}{\sqrt{x}} \) at \( x = 4 \).

When \( x = 4 \), \( y = \frac{1}{\sqrt[4]{4}} = \frac{1}{2} \), so the point of contact is \((4, \frac{1}{2})\).

Now \( \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} \). \( \therefore \) at \( x = 4 \), \( \frac{dy}{dx} = -\frac{1}{2} \left( 4^{-\frac{3}{2}} \right) = -\frac{1}{2} \left( \frac{1}{8} \right) = -\frac{1}{16} \)

\( \therefore \) the normal at \((4, \frac{1}{2})\) has gradient 16.

So the equation is \( \frac{y - \frac{1}{2}}{x - 4} = 16 \)

\[
\therefore \quad y - \frac{1}{2} = 16x - 64
\]

\[
\therefore \quad y = 16x - \frac{127}{2}
\]

---

2. a. The tangent shown on the graph passes through \((0, 5)\) and \((5, 0)\).

\( \therefore \) the gradient of the tangent is \( \frac{0 - 5}{5 - 0} = -1 \), so \( f'(5) = -1 \).

Also, since the tangent passes through \((0, 5)\), it has equation \( \frac{y - 5}{x - 0} = -1 \)

\[
\therefore \quad y - 5 = -x
\]

\[
\therefore \quad y = -x + 5
\]

So when \( x = 3 \), \( y = -3 + 5 = 2 \)

\( \therefore \) the point of contact is \((3, 2)\), and hence \( f(3) = 2 \).

b. \( f(x) \) has the form \( f(x) = ax^2 + bx + c \)

The \( y \)-intercept is 14. \( \therefore \) \( f(0) = 14 \)

\[
\therefore \quad a(0)^2 + b(0) + c = 14
\]

\[
\therefore \quad c = 14
\]

Also, \( f'(3) = 2 \)

\[
\therefore \quad a(3)^2 + b(3) + 14 = 2
\]

\[
\therefore \quad 9a + 3b = -12 \quad \ldots \quad (1)
\]

\[
\therefore \quad f'(3) = -1
\]

and \( f''(x) = 2ax + b \)

\[
\therefore \quad 2a(3) + b = -1
\]

\[
\therefore \quad 6a + b = -1
\]

\[
\therefore \quad b = -6a - 1 \quad \ldots \quad (2)
\]

Substituting (2) into (1),

\[
9a + 3(-6a - 1) = -12
\]

\[
\therefore \quad 9a - 18a - 3 = -12
\]

\[
\therefore \quad -9a = -9
\]

\[
\therefore \quad a = 1
\]

Using (2), \( b = -6(1) - 1 \)

\[
\therefore \quad b = -7
\]

So, \( f(x) = x^2 - 7x + 14 \)
3. \( y = x^3 + ax + b \) \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2 + a \\
\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = 3 + a \\
\text{The equation of the tangent at } x = 1 \text{ is } y = 2x, \text{ so the gradient is 2.} \\
\therefore 3 + a = 2 \quad \text{and so } a = -1 \\
\text{Also at } x = 1, \text{ the tangent touches the curve.} \\
\therefore x^3 + ax + b = 2x \quad \text{when } x = 1 \\
\therefore (1)^3 + (-1)(1) + b = 2(1) \\
\therefore 1 - 1 + b = 2 \\
\therefore b = 2 \\

4. a. \( y = x^3 + ax^2 - 4x + 3 \) \quad \therefore \frac{dy}{dx} = 3x^2 + 2ax - 4 \\
The tangent at \( x = 1 \) is parallel to \( y = 3x \), so when \( x = 1, \quad \frac{dy}{dx} = 3 \\
\therefore 3 = 3(1)^2 + 2a(1) - 4 \\
\therefore 2a = 4 \\
\therefore a = 2 \\
\text{When } x = 1, \quad y = 1^3 + 2(1)^2 - 4(1) + 3 = 2 \\
\text{The contact point is } (1, 2) \text{ and since the gradient is 3, the tangent at } (1, 2) \text{ has equation} \\
\frac{y - 2}{x - 1} = 3 \\
\therefore y - 2 = 3x - 3 \\
\therefore y = 3x - 1 \\
b. \text{The tangent meets the curve where } x^3 + 2x^2 - 4x + 3 = 3x - 1 \\
\therefore x^3 + 2x^2 - 7x + 4 = 0 \\
\text{Since the line touches the curve at } x = 1, \quad (x - 1)^2 \text{ must be a factor.} \\
\text{Consequently, } x^3 + 2x^2 - 7x + 4 = (x - 1)^2(x + 4) = 0 \quad \{\text{since the constant term is 4}\} \\
\therefore \text{the curve cuts the tangent when } x = -4. \\
\text{When } x = -4, \quad y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13 \\
\therefore \text{the curve cuts the tangent at } (-4, -13). \\

5. \quad y = \ln(x^4 + 3) \\
\therefore \frac{dy}{dx} = \frac{4x^3}{x^4 + 3} \\
\therefore \text{when } x = 1, \quad \frac{dy}{dx} = \frac{4(1)^3}{1^4 + 3} = 1 \quad \text{and } \quad y = \ln(1^4 + 3) = \ln 4 \\
\therefore \text{the tangent has equation } \frac{y - \ln 4}{x - 1} = 1 \quad \text{or } \quad y = x - 1 + \ln 4 \\
\text{Now when } x = 0, \quad y = \ln 4 - 1, \text{ so the tangent cuts the } y\text{-axis at } (0, \ln 4 - 1). \\

6. a. \quad f(x) = 2x^3 - 3x^2 - 36x + 7 \\
\therefore f'(x) = 6x^2 - 6x - 36 \\
\quad = 6(x^2 - x - 6) \\
\quad = 6(x - 3)(x + 2) \quad \text{with sign diagram:} \\
\text{Now } f(-2) = 51, \quad f(3) = -74, \text{ so there is a local maximum at } (-2, 51), \text{ and a local} \\
\text{minimum at } (3, -74). \\
f''(x) = 12x - 6 \\
\quad = 6(2x - 1) \quad \text{with sign diagram:} \\
\text{Now } f\left(\frac{1}{2}\right) = -\frac{23}{2}, \text{ so there is a non-stationary inflection at } \left(\frac{1}{2}, -\frac{23}{2}\right).
\[ f(x) \] is increasing when \( x \leq -2 \) or \( x \geq 3 \),
and decreasing when \( -2 \leq x \leq 3 \).

\( f(x) \) is concave up when \( x \geq \frac{1}{2} \),
and concave down when \( x \leq \frac{1}{2} \).

7. Consider the normal to \( f(x) = \frac{3x}{1 + x} \) at \((2, 2)\).

\( f(x) \) is a quotient with \( u = 3x \) and \( v = 1 + x \)
\[ u' = 3 \quad \text{and} \quad v' = 1 \]
\[ f'(x) = \frac{3(1 + x) - 1(3x)}{(1 + x)^2} = \frac{3}{(1 + x)^2} \quad \{ \text{quotient rule} \} \]
\[ f'(2) = \frac{3}{9} = \frac{1}{3} \]
\( \therefore \) the normal at \((2, 2)\) has gradient \(-3\).

So, the equation of the normal is
\[ \frac{y - 2}{x - 2} = -3 \]
\[ \therefore y - 2 = -3(x - 2) \]
\[ \therefore y = -3x + 8 \]

When \( x = 0 \), \( y = 8 \) and when \( y = 0 \), \( x = \frac{8}{3} \)
\( \therefore \) B and C are at \((0, 8)\) and \(\left(\frac{8}{3}, 0\right)\),
and the distance \(BC = \sqrt{\left(0 - \frac{8}{3}\right)^2 + (8 - 0)^2} = \sqrt{\frac{64}{9} + 64} = \sqrt{\frac{512}{9}} = \frac{8\sqrt{16}}{3}\) units.

8. \( f(x) = x^3 - 4x^2 + 4x \)
a. \( f(0) = 0 \), so the y-intercept is 0.
\[ f(x) \] cuts the x-axis when \( y = 0 \)
\[ x(x - 2)^2 = 0 \]
\( \therefore \) the x-intercepts are 0 and 2.

b. \( f'(x) = 3x^2 - 8x + 4 \)
\( = (3x - 2)(x - 2) \)
which is 0 when \( x = \frac{2}{3} \) or 2

Sign diagram of \( f'(x) \):

Now \( f\left(\frac{2}{3}\right) = \frac{20}{27} \), so there is a local maximum at \(\left(\frac{2}{3}, \frac{20}{27}\right)\), and a local minimum at \((2, 0)\).

\( f''(x) = 6x - 8 = 2(3x - 4) \)

Sign diagram of \( f''(x) \):

Now \( f\left(\frac{2}{3}\right) = \frac{16}{27} \), so there is a non-stationary inflection at \(\left(\frac{4}{3}, \frac{16}{27}\right)\).

9. \( y = \frac{1}{\sin x} = (\sin x)^{-1} \)
When \( x = \frac{\pi}{3} \), \( y = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}} \)
\[ \therefore \frac{dy}{dx} = -(\sin x)^{-2}(\cos x) \]
\[ = -\frac{\cos x}{\sin^2 x} \]
\[ \therefore \text{the tangent has equation} \quad \frac{y - \frac{2}{\sqrt{3}}}{x - \frac{\pi}{3}} = -\frac{2}{3} \text{ which is} \]
\[ 3y - 2\sqrt{3} = -2x + \frac{2\pi}{3} \]
\[ \text{or} \quad 2x + 3y = 2\sqrt{3} + \frac{2\pi}{3} \]
\[ y = \cos\left(\frac{x}{2}\right) \quad \text{When} \quad x = \frac{\pi}{2}, \quad y = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \]
\[ \frac{dy}{dx} = -\frac{1}{2} \sin\left(\frac{x}{2}\right) \quad \text{and} \quad \frac{dy}{dx} = -\frac{1}{2} \sin\left(\frac{x}{4}\right) = -\frac{1}{2\sqrt{2}} \]
\[ \therefore \text{the normal has gradient} \ 2\sqrt{2}, \ \text{and its equation is} \quad \frac{y - \frac{1}{\sqrt{2}}}{x - \frac{\pi}{2}} = 2\sqrt{2} \]
\[ \therefore \quad y - \frac{1}{\sqrt{2}} = 2\sqrt{2}x - \pi\sqrt{2} \]
\[ \therefore \quad y - 2\sqrt{2}x = \frac{1}{\sqrt{2}} - \pi\sqrt{2} \]
\[ \text{or} \quad \sqrt{2}y - 4x = 1 - 2\pi \]

10
\[ f(x) = 3x^3 + ax^2 + b \]
\[ \therefore f'(x) = 9ax^2 + 2ax \]
Since the tangent at \((-2, 14)\) has gradient 0,
\[ f'(-2) = 0 \]
\[ \therefore 36 - 4a = 0 \]
\[ \therefore a = 9 \]
As the point \((-2, 14)\) lies on the curve,
\[ 14 = 3(-2)^3 + 9(-2)^2 + b \]
\[ \therefore b = 14 + 24 - 36 \]
\[ \therefore b = 2 \]
\[ \therefore f'(x) = 9x^2 + 18x \]
\[ \therefore f''(x) = 18x + 18 \quad \text{and so} \quad f''(-2) = -36 + 18 = -18 \]

11
The curves \[ y = \sqrt{3x + 1} \quad \text{and} \quad y = \sqrt{5x - x^2} \] meet when \[ \sqrt{3x + 1} = \sqrt{5x - x^2} \]
Squaring both sides,
\[ 3x + 1 = 5x - x^2 \]
\[ \therefore x^2 - 2x + 1 = 0 \]
\[ \therefore (x - 1)^2 = 0 \]
\[ \therefore x = 1 \]
When \( x = 1, \ y = \sqrt{3 + 1} = 2, \) so the curves meet at \((1, 2)\).
Now for \( y = \sqrt{3x + 1} = (3x + 1)^{\frac{1}{2}} \)
Check: \( y = \sqrt{5x - x^2} = (5x - x^2)^{\frac{1}{2}} \)
\[ \frac{dy}{dx} = \frac{1}{2}(3x + 1)^{-\frac{1}{2}} (3) \]
\[ \frac{dy}{dx} = \frac{1}{2}(5x - x^2)^{-\frac{1}{2}} (5 - 2x) = \frac{5 - 2x}{2\sqrt{5x - x^2}} \]
\[ \therefore \text{at} \ (1, 2), \ \frac{dy}{dx} = \frac{3}{4} \]
\[ \therefore \text{at} \ (1, 2), \ \frac{dy}{dx} = \frac{5 - 2}{2\sqrt{5 - 1}} = \frac{3}{4} \quad \checkmark \]
\[ \therefore \text{the curves have a common tangent at their point of intersection.} \]
The equation of the common tangent at \((1, 2)\) is
\[ \frac{y - 2}{x - 1} = \frac{3}{4} \]
\[ \therefore 4(y - 2) = 3(x - 1) \]
\[ \therefore 4y = 3x + 5 \]

12
\( a \)
\[ f(x) = x + \ln x \quad \text{is defined when} \quad x > 0 \]
\[ f'(x) = 1 + \frac{1}{x} = \frac{x + 1}{x} \quad \text{which has sign diagram:} \]
\[ \begin{array}{c}
0 \\
+ \quad \rightarrow \quad x
\end{array} \]
\[ \therefore \quad f'(x) \quad \text{is increasing for all} \quad x > 0. \]
\[ f''(x) = -\frac{1}{x^2} \quad \text{which has sign diagram:} \]
\[ \begin{array}{c}
0 \\
- \quad \rightarrow \quad x
\end{array} \]
\[ \therefore \quad f'(x) \quad \text{is concave down for all} \quad x > 0. \]
\[ f(x) = x + \ln x \]

\[ f'(1) = 1 + \frac{1}{1} = 2 \]

\[ \therefore \text{the normal has gradient } -\frac{1}{2} \]

\[ \therefore \text{the normal has equation } y - 1 = -\frac{1}{2}(x - 1) \]

\[ \therefore 2y - 2 = -x + 1 \]

\[ \therefore x + 2y = 3 \]

\[ \text{At } x = B, \quad f''(x) = 0 \quad \text{but } f'(x) \neq 0 \]

\[ \therefore \text{f(x) has a non-stationary inflection point at } x = B. \]

\[ f'(x) \text{ is above the x-axis for } x < A \text{ and } x > C, \quad \text{and below the x-axis for } A < x < C \]

\[ \therefore \text{f(x) is increasing for } x < A, \text{ decreasing for } A < x < C, \text{ then increasing for } x > C \]

\[ \therefore \text{f(x) has a local maximum at } x = A \text{ and a local minimum at } x = C. \]