1. SEQUENCES/SERIES

→ BASICS: There are two main types of sequence:

- ARITHMETIC: add by \( d \) each term. E.G: 1, 4, 7, 10, ...
- GEOMETRIC: multiply by \( r \) each term. E.G: 2, 6, 18, 54, ...

→ \( n^{th} \) TERM: We can an arithmetic sequence using a general term formula (or '\( n^{th} \) temi formula'):

### ARITHMETIC

\[ U_n = U_1 + (n-1)d \]

- \( U_n \): \( n^{th} \) term
- \( U_1 \): 1st term
- \( d \): common difference

**E.G:** 20\(^{th}\) term of -3, -1, 1, 3, 5, ...

\[ \text{Ans: } U_1 = -3, \ n = 20, \ d = 2 \]

\[ \text{So } U_{20} = -3 + (20 - 1) \times 2 = -3 + (19 \times 2) = 35 \]

### GEOMETRIC

\[ U^n = U_1 \cdot r^{n-1} \]

- \( U^n \): \( n^{th} \) term
- \( U_1 \): 1st term
- \( r \): (common ratio)

**E.G:** 6\(^{th}\) term of 2, 10, 50, ...

\[ \text{Ans: } U_1 = 2, \ n = 6, \ r = \frac{10}{2} = \frac{50}{10} = 5 \]

\[ \text{So } U_6 = 2 \cdot 5^{6-1} = 2 \cdot 5^5 = 2 \cdot 3125 = 6250 \]

**E.G:** 7\(^{th}\) term of 12, -6, 3, -\( \frac{3}{2} \), ...

\[ \text{Ans: } U_1 = 12, \ n = 7, \ r = -\frac{6}{12} = -\frac{1}{2} \]

\[ \text{So } U_7 = 12 \cdot \left(-\frac{1}{2}\right)^{7-1} = 12 \cdot \frac{1}{64} = \frac{12}{64} = \frac{3}{16} \]
SUM/SERIES

\[ S_n \text{ NOTATION} \Rightarrow S_n = U_1 + U_2 + U_3 + \ldots + U_n \]

\[ \text{'Sigma' NOTATION} \Rightarrow \sum_{k=1}^{n} U_k \Rightarrow \text{the sum of all } U_k \text{'s from 1 to } n \left[U_1 + U_2 + \ldots + U_n\right] \]

ARITHMETIC SUM \Rightarrow S_n = \frac{n}{2} (U_1 + U_n) \text{ or } S_n = \frac{n}{2} (2U_1 + (n-1)d)

\text{E.G.: Find sum of } 3+7+11+15 \ldots \text{ to 20 terms:}

\text{Ans: } n = 20, U_1 = 3, d = 4

\text{So, } S_{20} = \frac{20}{2} (3 + (20 - 1)4) = 10 (676) = 820 \text{ using 2nd formula}

\text{E.G. 2: Find sum of } 5+8+11+\ldots + 101 :

\text{Ans: } n = 32, U_1 = 5, U_n = 101

\text{So, } S_{32} = \frac{32}{2} (5 + 101) = 16 (106) = 1696 \text{ using 1st formula}

GEOMETRIC SUM \Rightarrow S_n = \frac{U_1 (r^n - 1)}{r - 1} \text{ or } S_n = \frac{U_1 (1-r^n)}{1-r}

\text{E.G.: Find sum of } 2+6+18+54+\ldots \text{ to 12 terms:}

\text{Ans: } U_1 = 2, r = 3, n = 12

\text{So, } S_{12} = \frac{2 (3^{12} - 1)}{3 - 1} = 531,440 \text{ using 1st formula}

INFINITE GEO. SUM \Rightarrow S = \frac{U_1}{1-r} \text{ when } -1 < r < 1

\text{E.G.: Show } 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 2

\text{Ans: Could be written } \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1}

r = \frac{1}{2}, U_1 = 1, \text{ so } \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2
1.2 - LOGS/EXPONENTIALS

EXPONENTS

The following rules are NOT in the formula booklet:

E.G. 1 ⇒ \( 5^4 \times 5^7 = 5^{11} \)
E.G. 2 ⇒ \( \frac{k^8}{k^3} = k^5 \)
E.G. 3 ⇒ \( 8 \times 2^6 = 2^3 \times 2^6 = 2^{9+3} \)
E.G. 4 ⇒ \( \frac{9}{27} = \frac{3^2}{3^3} = 3^{-1} \)
E.G. 5 ⇒ \( \left( \frac{4}{3} \right)^{-\frac{1}{3}} = \frac{3}{4} \)

→ \( a^m \times a^n = a^{m+n} \)
→ \( \frac{a^n}{a^m} = a^{m-n}, a \neq 0 \)
→ \( (a^n)^m = a^{m \times n} \)
→ \( (ab)^n = a^n b^n \)
→ \( \left( \frac{a^n}{b^n} \right) = \frac{a^n}{b^n}, b \neq 0 \)
→ \( a^0 = 1, a \neq 0 \)
→ \( a^{-n} = \frac{1}{a^n}, a \neq 0 \)

RATIONAL EXPONENTS ⇒ When powers are written as fractions.

→ \( a^{\frac{1}{2}} = \sqrt{a} \)
→ \( a^{\frac{1}{3}} = \sqrt[3]{a} \)

⇒ \( a^{\frac{m}{n}} = \sqrt[n]{a^m} \)
⇒ \( a^{\frac{n}{m}} = \sqrt[m]{a^n} \)

Also not in formula booklet

EXPANSION/Factoring ⇒ Combine the rules below, and ones above.

→ \( a(b+c) = ab + ac \)
→ \( (a+b)(c+d) = ac + ad + bc + bd \)
→ \( (a-b)(a+b) = a^2 - b^2 \)
→ \( (a+b)^2 = a^2 + 2ab + b^2 \)
→ \( (a-b)^2 = a^2 - 2ab + b^2 \)

E.G. 5 ⇒ \( \frac{20a^5}{4^a} = 5^{\frac{1}{4}} \)

E.G. 1 ⇒ \( 2^x (2^x + 1) = 2^{2x} + 2^x = 2^x + 2^x = 4 + 2^x \)
E.G. 2 ⇒ \( (3^x+2)(3^x+5) = 3^{2x} + 7 \cdot 3^x + 10 \)
E.G. 3 ⇒ \( 3^{n+2} + 3^n = 3^n (3^2 + 1) \)
E.G. 4 ⇒ \( 4^x - 25 = (2^x + 5)(2^x - 5) \)

EXPONENTIAL EQUATIONS ⇒ Use \( a^x = a^x \), then \( x = \frac{\text{bottom}}{\text{top}} \)

E.G. 1 ⇒ \( 2^x = 8, 2^3 = 2^3, x = 3 \)
E.G. 2 ⇒ \( 7^{x+3} = 343, 7^{x+1} = 7^3, x+1 = 3, x = 2 \)
E.G. 3 ⇒ \( 2^{2x+1} = 8^{1-x}, (2^3)^{2x+1} = (2^3)^{x}, 2^{2x+2} = 2^{3-2x}, 4x+2 = 3-3x, 7x = 1, x = \frac{1}{7} \)
E.G. 4 ⇒ \( 3 \times 2^{2x+1} = 24, 3 \times 2^x = 3 \times 2^3, x+1 = 3, x = 2 \)
**LOGARITHMS**

**'GENERAL RULE'**

\[ \text{If } b = a^x \text{ then } x = \log_a b \]

\[ \text{E.g.} \ 3^x = 81, \ x = \log_3 81 = 4 \]

**FIRST RULES**

\[ x = \log_a a^x \quad \text{and} \quad x = a^{\log_a x} \]

\[ \text{E.g.} 1 \Rightarrow \log_5 0.2 = \log_5 \left(\frac{1}{5}\right) = \log_5 5^{-1} = -1 \]

\[ \text{E.g.} 2 \Rightarrow \log_2 \left(\frac{1}{2}\right) = \log_2 2^{-1} = -\frac{1}{2} \]

**LAWS**

\[ \log_c A + \log_c B = \log_c (AB) \quad \text{E.g.} 1 \Rightarrow \log 5 + \log 3 = \log (3 \times 5) = \log 15 \]

\[ \log_c A - \log_c B = \log_c \left(\frac{A}{B}\right) \quad \text{E.g.} 2 \Rightarrow 2 \log 7 - 3 \log 2 = \log 49 - \log 8 = \log \left(\frac{49}{8}\right) \]

\[ n \log_c A = \log_c (A^n) \quad \text{E.g.} 3 \Rightarrow 2 \log 3 + 3 = \log (3^2) + \log (10^3) \]

\[ = \log 9 + \log 1000 = \log 9000 \]

\[ \text{E.g.} 4 \Rightarrow \log A = \log b + 2 \log c, \ \log A = \log b + \log c^2 = \log (bc^2) \quad \therefore A = bc^2 \]

**NOTE:** 'ln x' means \( \log_e x \) (\( e \) is the 'natural exponential' \( \approx 2.718 \ldots \))

\[ \text{E.g.} 5 \Rightarrow \ln e^2 = \log_e e^2 = 2 / \text{E.g.} 6 \Rightarrow e^{\ln 3} = e^2 = e^{\log_e 3} = 3 \]

**'TAKING LOGS' (of each side)**

We use the fact that if \( x = y \), then \( \log x = \log y \)

\[ \text{E.g.} 1 \Rightarrow 2^x = 30, \ \log 2^x = \log 30, \ x \log 2 = \log 30, \ x = \frac{\log 30}{\log 2} \]

**CHANGE OF BASE**

**RULE**

\[ \log_b a = \frac{\log_c a}{\log_c b} \]

\[ \text{E.g.} 1 \Rightarrow \log_2 9 = \frac{\log 9}{\log 2} \approx 3.17 \]

\[ \text{or } \quad \frac{\log 9}{\log_2 9} \approx 3.17 \]

**REAL WORLD Q's**

Q: Investment of $5000, 5.2% p.a. interest, how long until $20000?

Sol.: Sequence of \( U_n = 5000 \cdot (1.052)^{n-1} \), use \( 20000 = 5000 \cdot (1.052)^{n-1} \)

\[ (1.052)^n = 4, \ \log (1.052)^n = \log 4, \ n - 1 = \frac{\log 4}{\log 1.052} \approx 27.3 \text{ yrs} \]
BINOMIAL EXPANSION

INVESTIGATION

Expand the following:

\[(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2\]
\[(a + b)^3 = (a + b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3\]

MEANING

Binomial expansion refers to the study and analysis of patterns that are created when \((a+b)^n\) is expanded, with any \(n\) value.

COEFFICIENTS

These are the constants that multiply each term of the expansion, marked in red in the above examples. We will look at four methods for finding them:

1) Manual expansion
   This is what we did above, but this will start to become a difficult task when we get to \((a+b)^5\), and larger \(n\)'s.

2) Pascal’s triangle
   This triangle is created by simply adding the two numbers immediately above:
   \[
   \begin{array}{cccccccc}
   n=1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
   n=2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\
   n=3 & 1 & 3 & 3 & 1 & 1 & 1 & 1 \\
   n=4 & 1 & 4 & 6 & 4 & 1 & 1 & 1 \\
   n=5 & 1 & 5 & 10 & 5 & 1 & 1 & 1 \\
   n=6 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
   \end{array}
   \]
   E.G.1 \(\Rightarrow\) Expand \((a+b)^5\): We will use the \(n=5\) row.
   \[a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\]
   E.G.2 \(\Rightarrow\) Find coefficient for the \(x^3\) term in expansion of \((x + 3)^4\):
   Here, \(a=x\) & \(b=3\). So we use the \(6a^2b^2\) term: \(6x^2(3^2) = 54x^2\)

3) Factorial Formula
   We can use the notation \(\binom{n}{r}\) or \(nCr\) to represent these coefficients, where \(n\) is the power, and \(r\) is the position in the expansion.
   Then there is a formula: \(\binom{n}{r} = \frac{n!}{r!(n-r)!}\) (starting at 0)
   E.G.1 \(\Rightarrow\) \(\binom{5}{3} = \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = 10\)
   You can check this by looking at the relevant section of Pascal's triangle.
4 GDC use

If this is a paper 2 question, you work out this \( nC_r \) value by simply finding the \( nC_r \) button on your GDC.

IB Question Solving:

You will most likely be asked to find the coefficient of a specific term in an expansion. The coefficient will usually also be multiplied by the \( b^r \) value as well.

**Note:** There are \((n+1)\) terms in the expansion of \((a+b)^n\).

**Note:** The 'Binomial Theorem' sums up the whole expansion:

\[
(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \ldots + \binom{n}{n}a^0 b^n
\]

**E.G.1:** Write the first three terms of \((1+2x)^n\):

**L sol:**

\[
1^n + \binom{1}{1}1^{10}(2x)^1 + \binom{1}{2}1^9(2x)^2 = 1 + (11)(1)(2x) + (55)(1)(4x^2)
\]

\[
= 1 + 22x + 220x^2
\]

**E.G.2:** Write the fourth term of \((2x+5)^5\):

**L sol:**

\[
\binom{5}{3}(2x)^3(5)^2 = (455)(4096x^3)(25) = 232960000x^3
\]

**E.G.3:** **IB:** Consider the expansion \((x+3)^n\)

a) Write down the number of terms in this expansion

**L sol:** \(10+1 = 11\)

b) Find the term containing \(x^3\):

**L sol:** \(n = 10, r = 7 \Rightarrow \binom{10}{7}x^33^7 = (120)x^3(2187) = 262440x^3\)

**E.G.4:** **IB:** The 5th term in the expansion of \((a+b)^n\) is given by \(\binom{n}{4}p^6(2q)^4\):

a) Write down the value of \(n\): 10

b) Write down \(a\ & b\): \(p \& 2q\)

c) Write down an expression for the 6th term:

**L sol:**

\[
\binom{10}{5}p^5(2q)^5 = (252)(p^5)(32q^5) = 8064p^5q^5
\]