1. \(\{(1, 3), (2, 4), (3, 5), (4, 6)\}\) is a function since no two ordered pairs have the same x-coordinate.

2. \(\{(1, 3), (3, 2), (1, 7), (-1, 4)\}\) is not a function since two of the ordered pairs, \((1, 3)\) and \((1, 7)\), have the same x-coordinate.

3. \(\{(2, -1), (2, 0), (2, 3), (2, 11)\}\) is not a function since each ordered pair has the same x-coordinate.

4. \(\{(7, 6), (5, 6), (3, 6), (-4, 6)\}\) is a function since no two ordered pairs have the same x-coordinate.

5. \(\{(0, 0), (1, 0), (3, 0), (5, 0)\}\) is a function since no two ordered pairs have the same x-coordinate.

6. \(\{(0, 0), (0, -2), (0, 2), (0, 4)\}\) is not a function since each ordered pair has the same x-coordinate.

Each line cuts the graph no more than once, so it is a function.

2. Each line cuts the graph no more than once, so it is a function.

3. Each line cuts the graph no more than once, so it is a function.

4. Each line cuts the graph no more than once, so it is a function.

5. Each line cuts the graph no more than once, so it is a function.

6. Each line cuts the graph no more than once, so it is a function.

3. The graph of a straight line is not a function if the graph is a vertical line. So, it is not a function if it has the form \(x = a\) for some constant \(a\).

The vertical line through \(x = a\) cuts the graph at every point, so it is not a function.
4. \( x^2 + y^2 = 9 \) is the equation of a circle, centre \((0, 0)\) and radius 3.
Now \( x^2 + y^2 = 9 \)
\[ \Rightarrow \quad y^2 = 9 - x^2 \]
\[ \therefore \quad y = \pm \sqrt{9 - x^2} \]
For any value of \( x \) where \(-3 < x < 3\), \( y \) has two real values. Hence \( x^2 + y^2 = 9 \) is not a function.

**EXERCISE 2B**

1. \( f(0) = 3(0) + 2 = 2 \)
   \( f(2) = 3(2) + 2 = 8 \)
   \( f(-1) = 3(-1) + 2 = -1 \)
   \( f(-5) = 3(-5) + 2 = -13 \)
   \( f(-\frac{1}{3}) = 3(-\frac{1}{3}) + 2 = 1 \)

2. \( f(0) = 3(0) - 0^2 + 2 \)
   \( f(3) = 3(3) - 3^2 + 2 \)
   \( f(-3) = 3(-3) - (-3)^2 + 2 \)
   \( = 2 \)
   \( = 9 - 9 + 2 \)
   \( = -9 + 2 \)
   \( = 2 \)
   \( = -16 \)
   \( = \frac{9}{2} - \frac{9}{4} + 2 \)
   \( = \frac{17}{4} \)

3. \( g(1) = 1 - \frac{4}{1} = -3 \)
   \( g(4) = 4 - \frac{4}{4} = 3 \)
   \( g(-1) = -1 - \left(-\frac{4}{1}\right) = 3 \)
   \( g(-4) = -4 - \left(-\frac{4}{-4}\right) = -3 \)
   \( g(-\frac{1}{2}) = -\frac{1}{2} - \frac{4}{(-\frac{1}{2})} = -\frac{1}{2} + 8 = \frac{15}{2} \)

4. \( f(a) = 7 - 3a \)
   \( f(-a) = 7 - 3(-a) \)
   \( = 7 + 3a \)
   \( f(a + 3) = 7 - 3(a + 3) \)
   \( = 7 - 3a - 9 \)
   \( = -3a - 2 \)
   \( f(b - 1) = 7 - 3(b - 1) \)
   \( = 7 - 3b + 3 \)
   \( = 10 - 3b \)
   \( f(x + 2) = 7 - 3(x + 2) \)
   \( = 7 - 3x - 6 \)
   \( = 1 - 3x \)
   \( f(x + h) = 7 - 3(x + h) \)
   \( = 7 - 3x - 3h \)

5. \( F(x + 4) \)
   \( = 2(x + 4)^2 + 3(x + 4) - 1 \)
   \( = 2(x^2 + 8x + 16) + 3x + 12 - 1 \)
   \( = 2x^2 + 16x + 32 + 3x + 11 \)
   \( = 2x^2 + 19x + 43 \)
   \( F(-x) \)
   \( = 2(-x)^2 + 3(-x) - 1 \)
   \( = 2x^2 - 3x - 1 \)
   \( F(x^2 - 1) \)
   \( = 2(x^2 - 1)^2 + 3(x^2 - 1) - 1 \)
   \( = 2(x^4 - 2x^2 + 1) + 3x^2 - 3 - 1 \)
   \( = 2x^4 - 4x^2 + 2 + 3x^2 - 4 \)
   \( = 2x^4 - x^2 - 2 \)

6. \( F(2 - x) \)
   \( = 2(2 - x)^2 + 3(2 - x) - 1 \)
   \( = 2(4 - 4x + x^2) + 6 - 3x - 1 \)
   \( = 8 - 8x + 2x^2 + 5 - 3x \)
   \( = 2x^2 - 11x + 13 \)
   \( F(x^2) \)
   \( = 2(x^2)^2 + 3(x^2) - 1 \)
   \( = 2x^4 + 3x^2 - 1 \)
   \( F(x + h) \)
   \( = 2(x + h)^2 + 3(x + h) - 1 \)
   \( = 2(x^2 + 2xh + h^2) + 3x + 3h - 1 \)
   \( = 2x^2 + 4xh + 2h^2 + 3x + 3h - 1 \)
6 a $G(2) = \frac{2(2) + 3}{2 - 4} = \frac{7}{-2} = -\frac{7}{2}$

b $G(x) = \frac{2x + 3}{x - 4}$ is undefined when $x - 4 = 0$ hence $x = 4$

So, when $x = 4$, $G(x)$ does not exist.

c $G(x + 2) = \frac{2(x + 2) + 3}{(x + 2) - 4} = \frac{2x + 4 + 3}{x + 2 - 4} = \frac{2x + 7}{x - 2}$

d $G(x) = -3$, so $\frac{2x + 3}{x - 4} = -3$ hence $2x + 3 = -3(x - 4)$

$\therefore 2x + 3 = -3x + 12$

$\therefore 5x = 9$ and so $x = \frac{9}{5}$

7 $f$ is the function which converts $x$ into $f(x)$ whereas $f(x)$ is the value of the function at any value of $x$.

8 a $V(4) = 9650 - 860(4) = 9650 - 3440 = 6210$

The value of the photocopyer 4 years after purchase is 6210 euros.

b If $V(t) = 5780$, then $9650 - 860t = 5780$

$\therefore 860t = 3870$

$\therefore t = 4.5$

The value of the photocopyer is 5780 euros after $4\frac{1}{2}$ years.

c Original purchase price is when $t = 0$,

$V(0) = 9650 - 860(0) = 9650$

The original purchase price was 9650 euros.

9 First sketch the linear function which passes through the two points (2, 1) and (5, 3).

Then sketch two quadratic functions which also pass through the two points.

10 $f(x) = ax + b$ where $f(2) = 1$ and $f(-3) = 11$

So, $a(2) + b = 1$ and $a(-3) + b = 11$

$\therefore 2a + b = 1$ and $-3a + b = 11$

$\therefore b = 1 - 2a \quad \ldots \quad (1)$ and $b = 11 + 3a \quad \ldots \quad (2)$

Solving (1) and (2) simultaneously, $1 - 2a = 11 + 3a$

$\therefore 5a = -10$

$\therefore a = -2$

Substituting $a = -2$ into (1) gives $b = 1 - 2(-2) = 5$. So, $a = -2, b = 5$

Hence $f(x) = -2x + 5$
11 \( f(x) = ax + \frac{b}{x} \) where \( f(1) = 1 \) and \( f(2) = 5 \)

So, \( a(1) + \frac{b}{1} = 1 \) and \( a(2) + \frac{b}{2} = 5 \)

\[ \therefore a + b = 1 \quad \text{and} \quad 2a + \frac{b}{2} = 5 \]

\[ \therefore a = 1 - b \quad \ldots (1) \]

Substituting (1) into (2), \( 2(1 - b) + \frac{b}{2} = 5 \)

\[ \therefore 2 - 2b + \frac{b}{2} = 5 \]

\[ \therefore - \frac{3b}{2} = 3 \]

\[ \therefore b = -2 \]

Substituting \( b = -2 \) into (1) gives \( a = 1 - (-2) = 3. \)

So, \( a = 3, \ b = -2. \)

12 \( T(x) = ax^2 + bx + c \) where \( T(0) = -4, \ T(1) = -2, \) and \( T(2) = 6 \)

So, \( a(0)^2 + b(0) + c = -4 \)

\[ \therefore c = -4 \]

Also, \( a(1)^2 + b(1) + c = -2 \) and \( a(2)^2 + b(2) + c = 6 \)

\[ \therefore a + b + c = -2 \quad \text{and} \quad 4a + 2b + c = 6 \]

Substituting \( c = -4 \) into both equations gives

\[ a + b + (-4) = -2 \quad \text{and} \quad 4a + 2b + (-4) = 6 \]

\[ \therefore a + b = 2 \quad \text{and} \quad 4a + 2b = 10 \quad \ldots (2) \]

\[ \therefore a = 2 - b \quad \ldots (1) \]

Substituting (1) into (2) gives \( 4(2 - b) + 2b = 10 \)

\[ \therefore 8 - 4b + 2b = 10 \]

\[ \therefore -2b = 2 \]

\[ \therefore b = -1 \]

Substituting \( b = -1 \) into (1) gives \( a = 2 - (-1) = 3. \)

\[ \therefore a = 3, \ b = -1, \) and \( c = -4. \) So, \( T(x) = 3x^2 - x - 4. \)

**EXERCISE 2C**

1 a Domain is \( \{x \mid x \geq -1\} \)
Range is \( \{y \mid y \leq 3\} \)

c Domain is \( \{x \mid x \neq 2\} \)
Range is \( \{y \mid y \neq -1\} \)

e Domain is \( \{x \mid x \in \mathbb{R}\} \)
Range is \( \{y \mid y \geq -1\} \)

\( f(x) \) is defined when \( x + 6 \geq 0 \)

\[ \therefore f(x) \text{ is defined for } x \geq -6 \]

\[ \therefore \text{ the domain is } \{x \mid x \geq -6\}. \]

g Domain is \( \{x \mid x \geq -4\} \)
Range is \( \{y \mid y \leq -2\} \)

\( f(x) \) is defined when \( 3 - 2x > 0 \)

\[ \therefore f(x) \text{ is defined for } x < \frac{3}{2} \]

\[ \therefore \text{ the domain is } \{x \mid x < \frac{3}{2}\}. \]

b Domain is \( \{x \mid -1 < x \leq 5\} \)
Range is \( \{y \mid 1 < y \leq 3\} \)

d Domain is \( \{x \mid x \in \mathbb{R}\} \)
Range is \( \{y \mid 0 < y \leq 2\} \)

\( f(x) \) is defined when \( a^2 \neq 0 \)

\[ \therefore f(x) \text{ is defined for } x \neq 0 \]

\[ \therefore \text{ the domain is } \{x \mid x \neq 0\}. \]

f Domain is \( \{x \mid x \in \mathbb{R}\} \)
Range is \( \{y \mid y \leq 6 \frac{1}{4}\} \) or \( \{y \mid y \leq 2 \frac{23}{4}\} \)

h Domain is \( \{x \mid x \in \mathbb{R}\} \)
Range is \( \{y \mid y > -2\} \)
3 a. \( y = 2x - 1 \) can take any \( x \)-value and any \( y \)-value.
   
   \( \therefore \) the domain is \( \{ x \mid x \in \mathbb{R} \} \) and the range is \( \{ y \mid y \in \mathbb{R} \} \).

b. \( y = 3 \) can take any value of \( x \), but the only permissible value for \( y \) is 3.
   
   \( \therefore \) the domain is \( \{ x \mid x \in \mathbb{R} \} \) and the range is \( \{ 3 \} \).

c. \( y = \sqrt{x} \) is defined when \( x \geq 0 \), and a square root cannot be negative.
   
   \( \therefore \) the domain is \( \{ x \mid x \geq 0 \} \) and the range is \( \{ y \mid y \geq 0 \} \).

d. \( y = \frac{1}{x+1} \) is defined when \( x + 1 \neq 0 \), or when \( x \neq -1 \).

   \( y = \frac{1}{x+1} \) cannot be 0 for any value of \( x \).

   \( \therefore \) the domain is \( \{ x \mid x \neq -1 \} \) and the range is \( \{ y \mid y \neq 0 \} \).

e. \( y = -\frac{1}{\sqrt{x}} \) is defined when \( x > 0 \).

   If \( x \) is always positive, then \( y = -\frac{1}{\sqrt{x}} \) is always negative.

   \( \therefore \) the domain is \( \{ x \mid x > 0 \} \) and the range is \( \{ y \mid y < 0 \} \).

f. \( y = \frac{1}{3-x} \) is defined when \( 3-x \neq 0 \), or when \( x \neq 3 \).

   \( y = \frac{1}{3-x} \) cannot be 0 for any value of \( x \).

   \( \therefore \) the domain is \( \{ x \mid x \neq 3 \} \) and the range is \( \{ y \mid y \neq 0 \} \).

4 a. Domain is \( \{ x \mid x \geq 2 \} \)

   Range is \( \{ y \mid y \geq 0 \} \)

b. Domain is \( \{ x \mid x \neq 0 \} \)

   Range is \( \{ y \mid y > 0 \} \)

c. Domain is \( \{ x \mid x \leq 4 \} \)

   Range is \( \{ y \mid y \geq 0 \} \)

d. Domain is \( \{ x \mid x \in \mathbb{R} \} \)

   Range is \( \{ y \mid y \geq -2\frac{1}{4} \} \)

e. Domain is \( \{ x \mid x \in \mathbb{R} \} \)

   Range is \( \{ y \mid y \geq 2 \} \)

f. Domain is \( \{ x \mid x \leq -2 \) or \( x \geq 2 \} \)

   Range is \( \{ y \mid y \geq 0 \} \)

g. Domain is \( \{ x \mid x \in \mathbb{R} \} \)

   Range is \( \{ y \mid y \leq \frac{25}{16} \} \)

h. Domain is \( \{ x \mid x \neq 0 \} \)

   Range is \( \{ y \mid y \leq -2 \) or \( y \geq 2 \} \)
EXERCISE 2D

1. a) \( (f \circ g)(x) = f(g(x)) \)
   \[= f(x + 2) \]
   \[= 2(x + 2) + 3 \]
   \[= 2x + 4 \]

   b) \( (g \circ f)(x) = g(f(x)) \)
   \[= g(2x + 3) \]
   \[= 1 - (2x + 3) \]
   \[= -2x - 2 \]

   c) \( (f \circ g)(-3) \)
   \[= 5 - 2(-3) \] \{from a\}
   \[= 11 \]

2. a) \( (f \circ f)(x) = f(f(x)) \)
   \[= f(2 + x) \]
   \[= 2 + (2 + x) \]
   \[= 4 + x \]

   b) \( (f \circ g)(x) = f(g(x)) \)
   \[= f(2 + x) \]
   \[= 2 + (3 - x) \]
   \[= 5 - x \]

   c) \( (g \circ f)(x) = g(f(x)) \)
   \[= g(2 + x) \]
   \[= 3 - (2 + x) \]
   \[= 1 - x \]

3. a) \( (g \circ g)(x) = g(g(x)) \)
   \[= g(2x - 7) \]
   \[= 5(2x - 7) - 7 \]
   \[= 25x - 35 - 7 \]
   \[= 25x - 42 \]

   b) \( (f \circ g)(1) = f(g(1)) \)
   \[= f(5(1) - 7) \]
   \[= f(-2) \]
   \[= \sqrt{6 - (-2)} \]
   \[= \sqrt{8} \]

   c) \( (g \circ f)(6) = g(f(6)) \)
   \[= g(\sqrt{6 - 6}) \]
   \[= g(0) \]
   \[= 5(0) - 7 \]
   \[= -7 \]

4. \( (f \circ g)(x) = f(g(x)) \)
   \[= f(2 - x) \]
   \[= (2 - x)^2 \]
   Domain is \( \{x \mid x \in \mathbb{R}\} \)
   Range is \( \{y \mid y \geq 0\} \)

   \( (g \circ f)(x) = g(f(x)) \)
   \[= g(x^2) \]
   \[= 2 - x^2 \]
   Domain is \( \{x \mid x \in \mathbb{R}\} \)
   Range is \( \{y \mid y \leq 2\} \)
5 a \( (f \circ g)(x) = f(g(x)) \)
\( = f(g(x)) \)
\( = f(3 - x) \)
\( = g(x^2 + 1) \)
\( = (3 - x)^2 + 1 \)
\( = 9 - 6x + x^2 + 1 \)
\( = x^2 - 6x + 10 \)

b \( (g \circ f)(x) = f(x) \)
\( = f(x) \)
\( = g(x^2) \)
\( = g(x^2 + 1) \)
\( = (x^2 + 1)^2 + 1 \)
\( = 3 - (x^2 + 1) \)
\( = 3 - x^2 - 1 \)
\( = 2 - x^2 \)

\( \therefore \quad 2 - x^2 = f(x) \) \{from \( \text{a} \) \( \text{ii} \) \}
\( \therefore \quad 2 - x^2 = x^2 + 1 \)
\( \therefore \quad 2x^2 = 1 \)
\( \therefore \quad x^2 = \frac{1}{2} \)
\( \therefore \quad x = \pm \sqrt{\frac{1}{2}} \)

6 a \( ax + b = cx + d \) is true for all \( x \) \{given\}
When \( x = 0 \), \( a(0) + b = c(0) + d \)
\( \therefore \quad b = d \) .... \(*\)
When \( x = 1 \), \( a(1) + b = c(1) + d \)
\( \therefore \quad a + b = c + d \)
But from \(*\), \( b = d \), so \( a + d = c + d \)
\( \therefore \quad a = c \)

b \( (f \circ g)(x) = x \) for all \( x \) \{given\}
\( \therefore \quad f(g(x)) = x \)
\( \therefore \quad f(ax + b) = x \)
\( \therefore \quad 2(ax + b) + 3 = x \)
\( \therefore \quad 2ax + 2b + 3 = x \) for all \( x \)
\( \therefore \quad 2a = 1 \) and \( 2b + 3 = 0 \) \{using \( \text{a} \) \}
\( \therefore \quad a = \frac{1}{2} \) and \( b = -\frac{3}{2} \)
So, \( a = \frac{1}{2} \) and \( b = -\frac{3}{2} \) as required.

c If \( (g \circ f)(x) = x \)
then \( g(f(x)) = x \)
\( \therefore \quad g(2x + 3) = x \)
\( \therefore \quad a(2x + 3) + b = x \)
\( \therefore \quad 2ax + 3a + b = x \)
\( \therefore \quad 2a = 1 \) and \( 3a + b = 0 \) \{using \( \text{a} \) \}
\( \therefore \quad a = \frac{1}{2} \) and \( b = -\frac{3}{2} \)
\( \therefore \quad \) the result in b is also true if \( (g \circ f)(x) = x \) for all \( x \).

7 a \( (f \circ g)(x) = f(g(x)) \)
\( = f(x^2) \)
\( = \sqrt{1 - x^2} \)
\( \therefore \quad \) \( (f \circ g)(x) = \sqrt{1 - x^2} \) is defined when \( 1 - x^2 \geq 0 \)
\( \therefore \quad x^2 \leq 1 \)
\( \therefore \quad -1 \leq x \leq 1 \)
\( \therefore \quad \) the domain is \( \{x \mid -1 \leq x \leq 1\} \)
and the range is \( \{y \mid 0 \leq y \leq 1\} \)

EXERCISE 2E

1 a \( \frac{-1}{2} + x \)
\( \frac{-1}{3} + x \)
\( \frac{1}{0} - x \)
\( \frac{-1}{2} - x \)
\( \frac{-1}{0} + x \)
\( \frac{-1}{2} + x \)
\( \frac{-1}{2} - x \)

b \( \frac{-1}{2} + x \)
\( \frac{-1}{2} + x \)
\( \frac{1}{0} - x \)
\( \frac{-1}{2} - x \)
\( \frac{-1}{2} + x \)
\( \frac{-1}{2} - x \)
\( \frac{-1}{2} + x \)

2 a \( y = (x + 4)(x - 2) \) is zero when \( x = -4 \) or \( 2 \).
When \( x = 0 \), \( y = (4)(-2) = -8 < 0 \).
The factors are single, so the signs alternate.
\( \therefore \quad \) sign diagram is: \( \frac{-1}{4} - \frac{1}{2} + x \)

b \( y = x(x - 3) \) is zero when \( x = 0 \) or \( 3 \).
When \( x = 10 \), \( y = 10(7) = 70 > 0 \).
The factors are single, so the signs alternate.
\( \therefore \quad \) sign diagram is: \( \frac{1}{0} - \frac{1}{3} + x \)
\[ e \quad y = x(x + 2) \text{ is zero when } x = -2 \text{ or } 0. \]
\[ \text{When } x = 10, \quad y = 10(12) = 120 > 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{2} \quad + \quad 0 \quad + \quad \frac{1}{2} \quad - \quad x \]

\[ d \quad y = -(x + 1)(x - 3) \text{ is zero when } x = -1 \text{ or } 3. \]
\[ \text{When } x = 0, \quad y = -(1)(-3) = 3 > 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{3} \quad + \quad 0 \quad + \quad \frac{1}{3} \quad - \quad x \]

\[ e \quad y = (2x - 1)(3 - x) \text{ is zero when } x = \frac{1}{2} \text{ or } 3. \]
\[ \text{When } x = 0, \quad y = (1)(-3) = -3 < 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{3} \quad + \quad 0 \quad + \quad \frac{1}{3} \quad - \quad x \]

\[ f \quad y = (5 - x)(1 - 2x) \text{ is zero when } x = \frac{1}{2} \text{ or } 5. \]
\[ \text{When } x = 0, \quad y = (5)(1) = 5 > 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{5} \quad + \quad 0 \quad + \quad \frac{1}{5} \quad - \quad x \]

\[ g \quad y = x^2 - 9 = (x + 3)(x - 3) \text{ is zero when } x = -3 \text{ or } 3. \]
\[ \text{When } x = 0, \quad y = (3)(-3) = -9 < 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{3} \quad + \quad 0 \quad + \quad \frac{1}{3} \quad - \quad x \]

\[ h \quad y = 4 - x^2 = (2 + x)(2 - x) \text{ is zero when } x = -2 \text{ or } 2. \]
\[ \text{When } x = 0, \quad y = (2)(2) = 4 > 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{2} \quad - \quad 0 \quad + \quad \frac{1}{2} \quad - \quad x \]

\[ i \quad y = 5x - x^2 = x(5 - x) \text{ is zero when } x = 0 \text{ or } 5. \]
\[ \text{When } x = 10, \quad y = 10(-5) = -50 < 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{5} \quad + \quad 0 \quad + \quad \frac{1}{5} \quad - \quad x \]

\[ j \quad y = 3x - 3x^2 = x(3 - x) \text{ is zero when } x = 0 \text{ or } 1. \]
\[ \text{When } x = 0, \quad y = (-1)(1) = -2 > 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{3} \quad - \quad 0 \quad + \quad \frac{1}{3} \quad - \quad x \]

\[ k \quad y = 2 - 8x = 2(1 + 2x)(1 - 2x) \text{ is zero when } x = -\frac{1}{2} \text{ or } \frac{1}{2}. \]
\[ \text{When } x = 0, \quad y = 2(1)(1) = 2 > 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{2} \quad + \quad 0 \quad + \quad \frac{1}{2} \quad - \quad x \]

\[ l \quad y = 6 - 16x - 6x^2 = 2(3 + x)(1 - 3x) \text{ is zero when } x = -3 \text{ or } \frac{1}{3}. \]
\[ \text{When } x = 0, \quad y = 2(3)(1) = 6 > 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{3} \quad + \quad 0 \quad + \quad \frac{1}{3} \quad - \quad x \]

\[ m \quad y = -15x^2 - x + 2 = (5x + 2)(1 - 3x) \text{ is zero when } x = -\frac{2}{5} \text{ or } \frac{1}{3}. \]
\[ \text{When } x = 0, \quad y = (2)(1) = 2 > 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{3} \quad + \quad 0 \quad + \quad \frac{1}{3} \quad - \quad x \]

\[ n \quad y = -2x^2 + 9x + 5 = (2x + 1)(5 - x) \text{ is zero when } x = -\frac{1}{2} \text{ or } 5. \]
\[ \text{When } x = 0, \quad y = (1)(5) = 5 > 0. \]
\[ \text{The factors are single, so the signs alternate.} \]
\[ \therefore \text{sign diagram is:} \quad \frac{-1}{5} \quad - \quad 0 \quad + \quad \frac{1}{5} \quad - \quad x \]

3 \( a \quad y = (x + 2)^2 \) is zero when \( x = -2 \).
\[ \text{When } x = 0, \quad y = 2^2 = 4 > 0. \]
\[ \text{The factor is squared, so the sign does not change.} \]
\[ \therefore \text{sign diagram is:} \quad + \quad 0 \quad + \quad -2 \quad + \quad x \]

3 \( b \quad y = (x - 3)^2 \) is zero when \( x = 3 \).
\[ \text{When } x = 0, \quad y = (-3)^2 = 9 > 0. \]
\[ \text{The factor is squared, so the sign does not change.} \]
\[ \therefore \text{sign diagram is:} \quad + \quad 0 \quad + \quad 3 \quad + \quad x \]
c  \( y = -(x + 2)^2 \) is zero when \( x = -2 \).
When \( x = 0 \), \( y = -(2^2) = -4 < 0 \).
The factor is squared, so the sign does not change.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
-2
\end{array} \rightarrow x \]

d  \( y = -(x - 4)^2 \) is zero when \( x = 4 \).
When \( x = 0 \), \( y = -(-4)^2 = -16 < 0 \).
The factor is squared, so the sign does not change.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
-4
\end{array} \rightarrow x \]

e  \( y = x^2 - 2x + 1 = (x - 1)^2 \) is zero when \( x = 1 \).
When \( x = 0 \), \( y = (-1)^2 = 1 > 0 \).
The factor is squared, so the sign does not change.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
1
\end{array} \rightarrow x \]

f  \( y = -x^2 + 4x - 4 = -(x - 2)^2 \) is zero when \( x = 2 \).
When \( x = 0 \), \( y = -(-2)^2 = -4 < 0 \).
The factor is squared, so the sign does not change.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
-2
\end{array} \rightarrow x \]

g  \( y = 4x^2 - 4x + 1 = (2x - 1)^2 \) is zero when \( x = \frac{1}{2} \).
When \( x = 0 \), \( y = (-1)^2 = 1 > 0 \).
The factor is squared, so the sign does not change.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
\frac{1}{2}
\end{array} \rightarrow x \]

h  \( y = -x^2 - 6x - 9 = -(x + 3)^2 \) is zero when \( x = -3 \).
When \( x = 0 \), \( y = -(3^2) = -9 < 0 \).
The factor is squared, so the sign does not change.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
-3
\end{array} \rightarrow x \]

i  \( y = -4x^2 + 12x - 9 = -(2x - 3)^2 \) is zero when \( x = \frac{3}{2} \).
When \( x = 0 \), \( y = -(3^2) = -9 < 0 \).
The factor is squared, so the sign does not change.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
\frac{3}{2}
\end{array} \rightarrow x \]

4  a  \( y = \frac{x + 2}{x - 1} \) is zero when \( x = -2 \) and
undefined when \( x = 1 \).
When \( x = 0 \), \( y = \frac{2}{-1} = -2 < 0 \).
Since the factors are single, the signs alternate.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
-2
\end{array} \rightarrow x \]

c  \( y = \frac{2x + 3}{4 - x} \) is zero when \( x = -3 \) and
undefined when \( x = 4 \).
When \( x = 0 \), \( y = \frac{3}{4} > 0 \).
Since the factors are single, the signs alternate.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
-3
\end{array} \rightarrow x \]

d  \( y = \frac{3x}{x - 2} \) is zero when \( x = 0 \) and
undefined when \( x = 2 \).
When \( x = 5 \), \( y = \frac{15}{3} = 5 > 0 \).
Since the factors are single, the signs alternate.
\[ \therefore \text{sign diagram is: } \quad \begin{array}{c}
\downarrow \\
0
\end{array} \rightarrow x \]
\( y = \frac{(x-1)^2}{x} \) is zero when \( x = 1 \) and undefined when \( x = 0 \).

When \( x = 2 \), \( y = \frac{4}{2} = 2 > 0 \).
Since the \((x-1)\) factor is squared, the sign does not change at \( x = 1 \).

\[ \therefore \text{ sign diagram is: } \begin{array}{c|c|c|c|c} 0 & 1 & + & \text{sign does not change at } x = 1. \end{array} \]

\( y = \frac{(x+2)(x-1)}{3-x} \) is zero when \( x = -2 \) or 1 and undefined when \( x = 3 \).
When \( x = 0 \), \( y = \frac{2(-1)}{3} = -\frac{2}{3} < 0 \).
Since the factors are single, the signs alternate.

\[ \therefore \text{ sign diagram is: } \begin{array}{c|c|c|c|c} -2 & 1 & + & - \end{array} \]

\( y = \frac{x^2-4}{-x} = \frac{(x-2)(x+2)}{-x} \) is zero when \( x = \pm 2 \) and undefined when \( x = 0 \).
When \( x = 1 \), \( y = \frac{-1(3)}{-1} = 3 > 0 \).
Since the factors are single, the signs alternate.

\[ \therefore \text{ sign diagram is: } \begin{array}{c|c|c|c|c} -2 & 0 & 2 & \end{array} \]

\( y = \frac{3-x}{2x^2-x-6} = \frac{3-x}{(2x+3)(x-2)} \) is zero when \( x = 3 \) and undefined when \( x = -\frac{3}{2} \) or 2.
When \( x = 0 \), \( y = \frac{3}{-3} = -1 < 0 \).
Since the factors are single, the signs alternate.

\[ \therefore \text{ sign diagram is: } \begin{array}{c|c|c|c|c} -\frac{3}{2} & 0 & \frac{3}{2} & \end{array} \]

**EXERCISE 2F**

1. \( f: x \mapsto \frac{3}{x-2} \) is undefined when \( x = 2 \), so \( x = 2 \) is a vertical asymptote.

   As \( |x| \to \infty \), \( f(x) \to 0 \), so \( y = 0 \) is a horizontal asymptote.

   II Domain is \( \{x \mid x \neq 2\} \), Range is \( \{y \mid y \neq 0\} \)

   III \( f(0) = \frac{3}{0-2} = -\frac{3}{2} \)

   So, the \( y \)-intercept is \( -\frac{3}{2} \).

   \( f(x) = 0 \) when \( \frac{3}{x-2} = 0 \),

   which has no solutions.

   \[ \therefore \text{ there is no } x \text{-intercept}. \]

   IV As \( x \to 2^- \), \( y \to -\infty \).

   As \( x \to 2^+ \), \( y \to \infty \).

   As \( x \to \infty \), \( y \to 0^+ \).

   As \( x \to -\infty \), \( y \to 0^- \).

b. \( f(x) = 2 - \frac{3}{x+1} \) is undefined when \( x = -1 \), so \( x = -1 \) is a vertical asymptote.

   As \( |x| \to \infty \), \( \frac{3}{x+1} \to 0 \), so \( f(x) \to 2 \) \( \therefore y = 2 \) is a horizontal asymptote.

   II Domain is \( \{x \mid x \neq -1\} \), Range is \( \{y \mid y \neq 2\} \)
III \( f(0) = 2 - \frac{3}{0 + 1} = -1 \)

So, the \( y \)-intercept is -1.

\( f(x) = 0 \) when \( 2 - \frac{3}{x+1} = 0 \)

\[ \therefore \frac{3}{x+1} = 2 \]

\[ \therefore x + 1 = \frac{3}{2} \]

\[ \therefore x = \frac{1}{2} \]

So, the \( x \)-intercept is \( \frac{1}{2} \).

IV As \( x \to -1^- \), \( y \to \infty \).

As \( x \to -1^+ \), \( y \to -\infty \).

As \( x \to \infty \), \( y \to 2^- \).

As \( x \to -\infty \), \( y \to 2^+ \).

c \( f : x \mapsto \frac{x + 3}{x - 2} \) is undefined when \( x = 2 \), so \( x = 2 \) is a vertical asymptote.

Now \( f(x) = \frac{x + 3}{x - 2} = \frac{1 + \frac{3}{x}}{1 - \frac{2}{x}} \)

\[ \therefore \text{as } |x| \to \infty, \ f(x) \to \frac{1}{1} = 1, \text{ and so } y = 1 \text{ is a horizontal asymptote.} \]

II Domain is \( \{x \mid x \neq 2\} \), Range is \( \{y \mid y \neq 1\} \)

III \( f(0) = \frac{0 + 3}{0 - 2} = -\frac{3}{2} \)

So, the \( y \)-intercept is \( -\frac{3}{2} \).

\( f(x) = 0 \) when \( \frac{x + 3}{x - 2} = 0 \)

\[ \therefore x + 3 = 0 \]

\[ \therefore x = -3 \]

So, the \( x \)-intercept is -3.

IV As \( x \to 2^- \), \( y \to -\infty \).

As \( x \to 2^+ \), \( y \to \infty \).

As \( x \to \infty \), \( y \to 1^+ \).

As \( x \to -\infty \), \( y \to 1^- \).

d \( f(x) = \frac{3x - 1}{x + 2} \) is undefined when \( x = -2 \), so \( x = -2 \) is a vertical asymptote.

\( f(x) = \frac{3x - 1}{x + 2} = \frac{3 - \frac{1}{x}}{1 + \frac{2}{x}} \)

As \( |x| \to \infty, \ f(x) \to \frac{3}{1} = 3 \) and so \( y = 3 \) is a horizontal asymptote.

II Domain is \( \{x \mid x \neq -2\} \), Range is \( \{y \mid y \neq 3\} \)

III \( f(0) = \frac{3(0) - 1}{0 + 2} = -\frac{1}{2} \)

So, the \( y \)-intercept is \( -\frac{1}{2} \).

\( f(x) = 0 \) when \( \frac{3x - 1}{x + 2} = 0 \)

\[ \therefore 3x - 1 = 0 \]

\[ \therefore x = \frac{1}{3} \]

So, the \( x \)-intercept is \( \frac{1}{3} \).
2 a The function is defined when \( cx + d \neq 0 \), or when \( x \neq -\frac{d}{c} \).
So, the domain is \( \{ x \mid x \neq -\frac{d}{c} \} \).

b The equation of the vertical asymptote is \( x = -\frac{d}{c} \).

c To find the horizontal asymptote of \( y = \frac{ax + b}{cx + d} \), we consider the function's behavior as \( |x| \to \infty \).
Now \( y = \frac{ax + b}{cx + d} = \frac{\alpha + \frac{b}{x}}{c + \frac{d}{x}} \).
\( \therefore \) as \( |x| \to \infty \), \( y \to \frac{\alpha}{c} \), and so \( y = \frac{\alpha}{c} \) is a horizontal asymptote.

**EXERCISE 2G**

1 a i

\[ f(x) \] passes through \((0, 1)\) and \((-\frac{1}{3}, 0)\).
\( \therefore f^{-1}(x) \) passes through \((1, 0)\) and \((0, -\frac{1}{3})\).

\( f^{-1}(x) \) has gradient \( \frac{-\frac{1}{3} - 0}{0 - 1} = \frac{-\frac{1}{3}}{-1} = \frac{1}{3} \).
So, its equation is \( y - 0 = \frac{1}{3} (x - 1) \).
which is \( y = \frac{x - 1}{3} \).
So, \( f^{-1}(x) = \frac{x - 1}{3} \).

b i

\[ f(x) \] passes through \((0, \frac{1}{2})\) and \((-2, 0)\).
\( \therefore f^{-1}(x) \) passes through \((\frac{1}{2}, 0)\) and \((0, -2)\).

\( f^{-1}(x) \) has gradient \( \frac{-2 - 0}{0 - \frac{1}{2}} = \frac{-2}{-\frac{1}{2}} = 4 \).
So, its equation is \( y - 0 = 4 \).
which is \( y = 4x - 2 \).
So, \( f^{-1}(x) = 4x - 2 \).
\[ f(x) \] passes through 
(0, 5) and \((-\frac{5}{2}, 0)\)
\[ : \quad f^{-1}(x) \] passes through (5, 0) and 
\((-\frac{5}{2}, 0)\).

\[ (f \circ f^{-1})(x) \quad \text{and} \quad (f^{-1} \circ f)(x) \]
\[ = f(f^{-1}(x)) \quad \text{and} \quad = f^{-1}(f(x)) \]
\[ = f(x) \quad \text{and} \quad = x \]

\[ f(x) \] passes through 
(0, \frac{3}{4}) and \((\frac{3}{2}, 0)\)
\[ : \quad f^{-1}(x) \] passes through \((\frac{3}{4}, 0)\) and 
\((0, \frac{3}{2})\).

\[ (f \circ f^{-1})(x) \quad \text{and} \quad (f^{-1} \circ f)(x) \]
\[ = f(f^{-1}(x)) \quad \text{and} \quad = f^{-1}(f(x)) \]
\[ = f(x) \quad \text{and} \quad = x \]

\[ f(x) \] passes through 
(0, 3) and \((-3, 0)\)
\[ : \quad f^{-1}(x) \] passes through (3, 0) and 
(0, -3).

\[ (f^{-1} \circ f)(x) = f^{-1}(f(x)) \quad \text{and} \quad (f \circ f^{-1})(x) = f(f^{-1}(x)) \]
\[ = f^{-1}(x + 3) \quad \text{and} \quad = f(x - 3) \]
\[ = (x + 3) - 3 \quad \text{and} \quad = (x - 3) + 3 \]
\[ = x \quad \text{and} \quad = x \]
4 a Domain of \( f(x) \) is \( \{x \mid -2 \leq x \leq 0\} \)  

b Range of \( f(x) \) is \( \{y \mid 0 \leq y \leq 5\} \)  

c Domain of \( f^{-1}(x) \) is \( \{x \mid 0 \leq x \leq 5\} \)  

d Range of \( f^{-1}(x) \) is \( \{y \mid -2 \leq y \leq 0\} \)  

5 a The functions in 3 e and 3 f are self-inverse functions.  

b Any linear function of the form \( y = a - x \) will be a self-inverse function, for example \( y = -x \) (where \( a = 0 \)):  

c Any rational function of the form \( y = \frac{a}{x} \) will be a self-inverse function, for example \( y = \frac{2}{x} \) (where \( a = 2 \)):  

6 Range of \( H^{-1}(x) \) is \( \{y \mid -2 \leq y < 3\} \)  

7 \( f \) is \( y = 2x - 5 \)  
\[ \therefore \text{ the inverse function is } x = 2y - 5 \]  
\[ \therefore 2y = x + 5 \]  
\[ \therefore y = \frac{x + 5}{2} \]  
\[ \therefore f^{-1}(x) = \frac{x + 5}{2}. \]  

So, \( f^{-1}(f^{-1}(x)) = f(x) \).  

To find \( f^{-1}(f^{-1}(x)) \), we need to find the inverse function for \( y = \frac{x + 5}{2} \)  
This is \( x = \frac{y + 5}{2} \)  
\[ \therefore 2x = y + 5 \]  
\[ \therefore y = 2x - 5 \]  
This is the original function \( f(x) \).
9 \( f(x) = \frac{1}{x} \) has inverse function \( x = \frac{1}{y} \) or \( y = \frac{1}{x} \)

So, \( f^{-1}(x) = \frac{1}{x} \), which means \( f(x) \) is a self-inverse function.

10 a \( f(x) = \frac{3x - 8}{x - 3} \) has graph

\[ f(x) = \frac{3x - 8}{x - 3} \]

The vertical line test shows it to be a function.
Symmetry about \( y = x \) shows it is a self-inverse function.

b \( f(x) = \frac{3x - 8}{x - 3} \) has inverse function \( x = \frac{3y - 8}{y - 3} \)

\[ \therefore x(y - 3) = 3y - 8 \]
\[ \therefore xy - 3x = 3y - 8 \]
\[ \therefore xy - 3y = 3x - 8 \]
\[ \therefore y(3 - x) = 3x - 8 \]
\[ \therefore y = \frac{3x - 8}{x - 3} \]

\[ \therefore f^{-1}(x) = \frac{3x - 8}{x - 3} \]

So, \( f(x) = f^{-1}(x) \), which means \( f(x) \) is a self-inverse function.

11 a \( f(x) = \frac{1}{2}x - 1 \) has inverse function \( x = \frac{1}{2}y - 1 \)

\[ \therefore x + 1 = \frac{1}{2}y \]
\[ \therefore y = 2x + 2 \quad \text{So,} \quad f^{-1}(x) = 2x + 2 \]

b \[ (f \circ f^{-1})(x) = f(f^{-1}(x)) \]
\[ = f(2x + 2) \quad \text{\{using a\}} \]
\[ = \frac{1}{2}(2x + 2) - 1 \]
\[ = x + 1 - 1 \]
\[ = x \]

\[ (f^{-1} \circ f)(x) = f^{-1}(f(x)) \]
\[ = \frac{3}{2}x - 1 \]
\[ = 2(\frac{3}{2}x - 1) + 2 \quad \text{\{using a\}} \]
\[ = x - 2 + 2 \]
\[ = x \]

12 a \( g \) is \( y = \frac{8 - x}{2} \)

so \( g^{-1} \) is \( x = \frac{8 - y}{2} \)

\[ \therefore 2x = 8 - y \]
\[ \therefore y = 8 - 2x \]

So, \( g^{-1}(x) = 8 - 2x \)

\[ \therefore g^{-1}(-1) = 8 - 2(-1) = 10 \]

b \( f \) is \( y = 2x + 5 \)

so \( f^{-1} \) is \( x = 2y + 5 \)

\[ \therefore 2y = x - 5 \]
\[ \therefore y = \frac{x - 5}{2} \]

So, \( f^{-1}(x) = \frac{x - 5}{2} \)

\[ \therefore f^{-1}(-3) = \frac{-3 - 5}{2} = \frac{-8}{2} = -4 \]

and \( g^{-1}(6) = 8 - 2 \times 6 = 8 - 12 = -4 \)

12 a \( f(g^{-1}(x)) = 9 \)

\[ \therefore f(g^{-1}(x)) = 9 \]
\[ \therefore f(8 - 2x) = 9 \]
\[ \therefore 2(8 - 2x) + 5 = 9 \]
\[ \therefore 16 - 4x + 5 = 9 \]
\[ \therefore -4x = -12 \]
\[ \therefore x = 3 \]
13  \( f \) is \( y = 5^2 \)
   so \( f(2) = 5^2 \)
   \[= 25\]

\( g \) is \( y = \sqrt{x} \) where \( x \geq 0 \)
   so \( g^{-1} \) is \( x = \sqrt{y} \) where \( x \geq 0 \)
   \[\therefore y = x^2\]
   \[\therefore g^{-1}(x) = x^2, \ x \geq 0\]
   \[\therefore g^{-1}(4) = 4^2\]
   \[= 16\]

\( f \) is \( y = 2x \)
so \( f^{-1} \) is \( x = 2y \)
\[\therefore y = \frac{x}{2}\]
\[\therefore f^{-1}(x) = \frac{x}{2}\]

\( g \) is \( y = 4x - 3 \)
so \( g^{-1} \) is \( x = 4y - 3 \)
\[\therefore 4y = x + 3\]
\[\therefore y = \frac{x + 3}{4}\]
\[\therefore g^{-1}(x) = \frac{x + 3}{4}\]

\((g \circ f)(x) = g(f(x))\)
\[= g(2x)\]
\[= 4(2x) - 3\]
\[= 8x - 3\]

\( (g \circ f)(x) = 8x - 3 \)
\[\therefore (g \circ f)^{-1} \) is \( x = 8y - 3 \)
\[\therefore y = \frac{x + 3}{8}\]

Now \( (f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) \)
\[= f^{-1} \left( \frac{x + 3}{4} \right) \]
\[= \left( \frac{x + 3}{4} \right) \]
\[= \frac{x + 3}{2}\]

\[\therefore (f^{-1} \circ g^{-1})(x) = \frac{x + 3}{8} = (g \circ f)^{-1}(x) \] as required

15  a  \( f \) is \( y = 2x \)
   so \( f^{-1} \) is \( x = 2y \)
   \[\therefore x \neq 2x\]
   So, \( f^{-1}(x) \neq f(x) \)

b  \( f \) is \( y = x \)
   so \( f^{-1} \) is \( x = y \)
   \[\therefore y = -x\]
   So, \( f^{-1}(x) = f(x) \)

b  \( f \) is \( y = -x \)
   so \( f^{-1} \) is \( x = y \)
   \[\therefore y = -x\]
   So, \( f^{-1}(x) = f(x) \)

e  \( f \) is \( y = x \)
   so \( f^{-1} \) is \( x = y \)
   \[\therefore y = -x\]
   So, \( f^{-1}(x) = f(x) \)

c  \( f \) is \( y = x \)
   so \( f^{-1} \) is \( x = y \)
   \[\therefore y = -x\]
   So, \( f^{-1}(x) = f(x) \)

d  \( f \) is \( y = \frac{2}{x} \)
   so \( f^{-1} \) is \( x = \frac{2}{y} \)
   \[\therefore x \neq \frac{2}{x}\]
   So, \( f^{-1}(x) \neq f(x) \)

e  \( f \) is \( y = \frac{2}{x} \)
   so \( f^{-1} \) is \( x = \frac{2}{y} \)
   \[\therefore y = -\frac{6}{x}\]
   So, \( f^{-1}(x) = f(x) \)

So, \( f^{-1}(x) = f(x) \) is true for parts b, c, d, and e.

16  a  If \( y = f(x) \) has an inverse function, then the inverse function must also be a function. It must satisfy the 'vertical line test', so no vertical line can cut it more than once. This condition for the inverse function cannot be satisfied if the original function does not satisfy the 'horizontal line test'. Thus, the 'horizontal line test' is a valid test for the existence of an inverse function.
This graph satisfies the 'horizontal line test' and therefore has an inverse function.

These graphs both fail the 'horizontal line test' so neither of these have inverse functions.

REVIEW SET 2A

1. a) Domain is \( \{ x \mid x \in \mathbb{R} \} \).
   Range is \( \{ y \mid y > -4 \} \).

Each line cuts the graph no more than once, so the graph shows a function.

b) Domain is \( \{ x \mid x \in \mathbb{R} \} \).
   Range is \( \{ 2 \} \).

Each line cuts the graph no more than once, so the graph shows a function.

c) Domain is \( \{ x \mid x \in \mathbb{R} \} \).
   Range is \( \{ y \mid y \leq -1 \text{ or } y \geq 1 \} \).

de) Domain is \( \{ x \mid x \in \mathbb{R} \} \).
   Range is \( \{ y \mid -5 \leq y \leq 5 \} \).

The lines cut the graph more than once, so the graph does not show a function.

2. \( f(x) = 2x - x^2 \)
   \( f(2) = 2(2) - 2^2 \)
   \( = 0 \)

\( f(3) = 2(-3) - (-3)^2 \)
\( = -6 - 9 \)
\( = -15 \)

\( f(-\frac{1}{2}) = 2(-\frac{1}{2}) - (-\frac{1}{2})^2 \)
\( = -1 - \frac{1}{4} \)
\( = -\frac{5}{4} \)

3. \( f(x) = ax + b \), where \( f(1) = 7 \) and \( f(3) = -5 \)

When \( f(1) = 7 \),
\( 7 = a(1) + b \)
\( 7 = a + b \)
\( \therefore a = 7 - b \) .... (1)

When \( f(3) = -5 \),
\( -5 = a(3) + b \)
\( -5 = 3a + b \)
\( \therefore -5 = 3(7 - b) + b \) \{using (1)\}
\( \therefore -5 = 21 - 3b + b \)
\( \therefore 2b = 26 \) and so \( b = 13 \)

Substituting \( b = 13 \) into (1), \( a = 7 - 13 = -6 \)

\( \therefore a = -6 \) and \( b = 13 \)

4. \( g(x) = x^2 - 3x \)

\( g(x + 1) = (x + 1)^2 - 3(x + 1) \)
\( = x^2 + 2x + 1 - 3x - 3 \)
\( = x^2 - x - 2 \)

\( g(x^2 - 2) = (x^2 - 2)^2 - 3(x^2 - 2) \)
\( = x^4 - 4x^2 + 4 - 3x^2 + 6 \)
\( = x^4 - 7x^2 + 10 \)
5 a i Domain is \( \{x \mid x \in \mathbb{R}\} \). Range is \( \{y \mid y \geq -5\} \).
ii \( x \)-intercepts are \(-1\) and \(5\), \( y \)-intercept is \(-\frac{25}{9}\).
iii The graph passes the 'vertical line test' so is therefore a function.

b i Domain is \( \{x \mid x \in \mathbb{R}\} \). Range is \( \{y \mid y = 1 \text{ or } -3\} \).
ii There are no \( x \)-intercepts, \( y \)-intercept is 1.
iii The graph passes the 'vertical line test' so is therefore a function.

6 a \( y = (3x + 2)(4 - x) \) is zero when \( x = -\frac{2}{3} \) or 4.
When \( x = 0 \), \( y = (2)(4) = 8 > 0 \).
Since the factors are single, the signs alternate.
\[\therefore \text{ sign diagram is } \begin{array}{ccc} - & + & - \\ -\frac{2}{3} & & 4 \end{array} \]

b \( y = \frac{x - 3}{x^2 + 4x + 4} = \frac{x - 3}{(x + 2)^2} \) is zero when \( x = 3 \) and undefined when \( x = -2 \).
When \( x = 0 \), \( y = \frac{-3}{4^2} = -\frac{3}{4} < 0 \).
Since the \( (x + 2) \) factor is squared, the sign does not change at \( x = -2 \)
\[\therefore \text{ sign diagram is } \begin{array}{ccc} - & - & + \\ -2 & & 3 \end{array} \]

7 \( f(x) = ax + b \)
Now \( f(2) = 1 \), so \( a(2) + b = 1 \)
\[\therefore b = 1 - 2a \quad \text{(*)} \]
Now \( f^{-1}(3) = 4 \), so \( f(4) = 3 \)
\[\therefore a(4) + b = 3 \]
\[\therefore 4a + (1 - 2a) = 3 \quad \{\text{from (*)}\} \]
\[\therefore 2a = 2 \]
\[\therefore a = 1 \]
Substituting \( a = 1 \) into (*), \( b = 1 - 2(1) = -1 \)
So, \( a = 1 \) and \( b = -1 \).

8 a

b

9 a \( f \) is \( y = 4x + 2 \)
\[\therefore f^{-1}(x) \text{ is } x = 4y + 2 \]
\[\therefore y = \frac{x - 2}{4} \]
\[\therefore f^{-1}(x) = \frac{x - 2}{4} \]

b \( f \) is \( y = \frac{3 - 5x}{4} \)
So \( f^{-1}(x) \) is \( x = \frac{3 - 5y}{4} \)
\[\therefore 4x = 3 - 5y \]
\[\therefore y = \frac{3 - 4x}{5} \]
\[\therefore f^{-1}(x) = \frac{3 - 4x}{5} \]
10  a  \[ f(x) = x^2 \quad b \quad (f \circ g)(-2) = f(g(-2)) \quad c \quad f(5) = 5^2 = 25 \]
\[ \therefore \quad f(-3) = (-3)^2 = 9 \quad \text{Now, } g(-2) = 1 - 6(-2) = 13 \quad \text{So, we need to find } x \text{ such that } 1 - 6x = 25 \]
\[ g(x) = 1 - 6x \]
\[ \therefore \quad g(-\frac{4}{3}) = 1 - 6(-\frac{4}{3}) = 1 + 8 = 9 \quad \therefore \quad (f \circ g)(-2) = f(13) = 13^2 = 169 \]
\[ \therefore \quad f(-3) = g(-\frac{4}{3}) \]
\[ 11 \quad f \text{ is } y = 3x + 6 \quad h \text{ is } y = \frac{x}{3} \]
so  \[ f^{-1}(x) = x = 3y + 6 \quad \text{so } h^{-1}(x) = x = \frac{y}{3} \]
\[ \therefore \quad y = \frac{x - 6}{3} \quad \therefore \quad y = 3x \]
\[ \therefore \quad f^{-1}(x) = \frac{x - 6}{3} \quad \therefore \quad h^{-1}(x) = 3x \]
Now \[ (f^{-1} \circ h^{-1})(x) = f^{-1}(h^{-1}(x)) = h(f(x)) = h(3x + 6) = \frac{3x + 6}{3} = x - 2 \]
so  \[ (f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = x = y + 2 \]
\[ \therefore \quad y = x - 2 \quad \therefore \quad (h \circ f)^{-1}(x) = x = 2 \]
\[ \therefore \quad (f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) \text{ as required.} \]

**REVIEW SET 2B**

1  a  \[ y = (x - 1)(x - 5) \]
\[ \therefore \quad \text{the } x\text{-intercepts are } x = 1 \text{ and } 5 \]
The vertex is at  \[ x = 3, \text{ with } y = (3 - 1)(3 - 5) = 2 \times (-2) = -4 \]
\[ \therefore \quad \text{the vertex is at } (3, -4) \]
The domain is  \{x \mid x \in \mathbb{R} \}. \quad \text{The range is } \{y \mid y \geq -4\}. \]

b  From the graph, the domain is  \{x \mid x \neq 0, x \neq 2 \} and the range is  \{y \mid y \leq -1 \text{ or } y > 0\}. 

2  a  \[ (f \circ g)(x) = f(g(x)) = f(x^2 + 2) = g(2x - 3) = g(2x^2 + 2 - 3) = 2x^2 + 4 - 3 = 2x^2 + 1 \]

b  \[ (g \circ f)(x) = g(f(x)) = g(2x - 3) = (2x - 3)^2 + 2 = 4x^2 - 12x + 9 + 2 = 4x^2 - 12x + 11 \]

3  a  \[ y = \frac{x^2 - 6x - 16}{x - 3} = \frac{(x + 2)(x - 8)}{x - 3} \]
\[ \text{is zero when } x = -2 \text{ or } 8 \text{ and undefined when } x = 3. \]
When  \[ x = 0, \quad y = \frac{-16}{-3} > 0. \]
Since the factors are single, the signs alternate. So, the sign diagram is:  \[ - \quad + \quad \frac{1}{3} \quad - \quad \frac{3}{8} \quad + \quad x \]

b  \[ y = \frac{x^2 + 9}{x + 5} + x \frac{(x + 5)}{(x + 5)} = \frac{x^2 + 6x + 9}{x + 5} = \frac{(x + 3)^2}{x + 5} \]
\[ \text{is zero when } x = -3 \]
and undefined when  \[ x = -5. \]
When  \[ x = 0, \quad y = \frac{9^2}{5} > 0. \] The \[ (x + 3) \] factor is squared, so the sign does not change at \[ x = -3. \]
So, the sign diagram is:  \[ - \quad + \quad \frac{1}{5} \quad + \quad \frac{3}{3} \quad + \quad x \]
4 a  \( f(x) = \frac{1}{x^2} \) is meaningless when \( x = 0 \).

b  ![Graph of \( f(x) = \frac{1}{x^2} \)]

c  Domain of \( f(x) \) is \( \{x \mid x \neq 0\} \).

Range of \( f(x) \) is \( \{y \mid y > 0\} \).

5 a  \( f(x) = \frac{ax + 3}{x - b} \) has asymptotes \( x = -1, \ y = 2 \).

\( f(x) \) is undefined when \( x - b = 0 \)

\[ x = b \] is the vertical asymptote.

But \( x = -1 \) is the vertical asymptote, so \( b = -1 \).

So, \[ f(x) = \frac{ax + 3}{x - (-1)} = \frac{ax + 3}{x + 1} = \frac{a + \frac{3}{x}}{1 + \frac{x}{x}} \]

As \( |x| \to \infty, \ f(x) \to a \) so the horizontal asymptote is \( y = a \).

But \( y = 2 \) is the horizontal asymptote, so \( a = 2 \).

\[ \therefore \ a = 2 \quad \text{and} \quad b = -1. \]

b  Domain of \( f \) is \( \{x \mid x \neq -1\} \) and range of \( f \) is \( \{y \mid y \neq 2\} \).

\[ \therefore \ \text{domain of } f^{-1} \text{ is } \{x \mid x \neq 2\} \quad \text{and range of } f^{-1} \text{ is } \{y \mid y \neq -1\}. \]

6 a  \( f : x \mapsto \frac{4x + 1}{2 - x} \) is undefined when \( x = 2 \).

\[ \therefore \ x = 2 \] is a vertical asymptote.

Now \[ f(x) = \frac{4x + 1}{2 - x} = \frac{4 + \frac{1}{x}}{-1 + \frac{2}{x}} \]

\[ \therefore \text{as } |x| \to \infty, \ f(x) \to \frac{-4}{1} = -4, \]

and so \( y = -4 \) is a horizontal asymptote.

d  \( f(0) = \frac{4(0) + 1}{2 - 0} = \frac{1}{2} \)

So, the \( y \)-intercept is \( \frac{1}{2} \).

\[ f(x) = 0 \text{ when } \frac{4x + 1}{2 - x} = 0 \]

\[ \therefore \ 4x + 1 = 0 \]

\[ \therefore \ x = -\frac{1}{4} \]

So, the \( x \)-intercept is \( -\frac{1}{4} \).

b  The domain is \( \{x \mid x \neq 2\} \).

The range is \( \{y \mid y \neq -4\} \).

c  As \( x \to 2^- \), \( y \to \infty \).

As \( x \to 2^+ \), \( y \to -\infty \).

As \( x \to \infty \), \( y \to -4^- \).

As \( x \to -\infty \), \( y \to -4^+ \).

[Graph of \( f(x) = \frac{4x + 1}{2 - x} \)]

7 a  \( (g \circ f)(x) = g(f(x)) \)

\[ = g(3x + 1) \]

\[ = \frac{2}{3x + 1} \]

b  \( (g \circ f)(x) = -4 \)

\[ \therefore \frac{2}{3x + 1} = -4 \]

\[ \therefore -4(3x + 1) = 2 \]

\[ \therefore -12x - 4 = 2 \]

\[ \therefore 12x = -6 \]

\[ \therefore x = -\frac{1}{2} \]
\[ h(x) = \frac{2}{3x + 1} \] is undefined when \[ 3x + 1 = 0 \] or \[ x = -\frac{1}{3}. \]
So, \( x = -\frac{1}{3} \) is a vertical asymptote.
As \( |x| \to \infty, \) \( h(x) \to 0 \)
\[ \therefore y = 0 \] is a horizontal asymptote.

III Range of \( h \) is \( \{ y \mid y \leq -\frac{4}{3} \text{ or } y \geq \frac{2}{3} \} \).

8 a

\[ f \circ f^{-1} \quad \text{and} \quad f^{-1} \circ f \]
\[ = f \left( f^{-1}(x) \right) \]
\[ = f \left( \frac{x + 7}{2} \right) \]
\[ = 2 \left( \frac{x + 7}{2} \right) - 7 \]
\[ = x + 7 - 7 \]
\[ = x \]
So, \( f \circ f^{-1} = f^{-1} \circ f = x \).

b The function \( f \) is \( y = 2x - 7 \)
so \( f^{-1} \) is \( x = 2y - 7 \)
\[ \therefore y = \frac{x + 7}{2} \]
So, \( f^{-1}(x) = \frac{x + 7}{2} \).

9 a

\[ y = f^{-1}(x) \]
\[ (1, -3) \]
\[ (2, -12) \]
\[ y = f(x) \]

b Range of \( f^{-1} \) is \( \{ y \mid 0 \leq y \leq 2 \} \).

1 a Domain is \( \{ x \mid x \geq -2 \} \). Range is \( \{ y \mid 1 \leq y < 3 \} \).

b Domain is \( \{ x \mid x \in \mathbb{R} \} \). Range is \( \{ y \mid y = -1, 1, \text{ or } 2 \} \).

2 a \( f(x) = x^2 + 3 \)
\[ \therefore f(-3) = (-3)^2 + 3 \]
\[ = 9 + 3 \]
\[ = 12 \]

b \( x^2 = 4 \)
\[ \therefore x = \pm 1 \]

3 a \( f(x) = 10 + \frac{3}{2x - 1} \)
is undefined when \( 2x - 1 = 0 \)
\[ \therefore x = \frac{1}{2} \]

b \( f(x) = \sqrt{x + 7} \)
is undefined when \( x + 7 < 0 \)
\[ \therefore x < -7 \]
4  \( a \quad f(x) = x(x + 4)(3x + 1) \) is zero when \( x = 0, -4, \) and \(-\frac{1}{3}\).
When \( x = 10, \ y = 10(14)(31) = 4340 > 0.\)
The factors are single so the signs alternate.
\[
\therefore \text{ sign diagram is: } \quad \begin{array}{c|c|c|c}
-4 & \frac{1}{3} & 0 \\
+ & + & +
\end{array}
\]

5  \( a \quad h(x) = 7 - 3x \)
\[
h(2x - 1) = 7 - 3(2x - 1)
= 7 - 6x + 3
= 10 - 6x
\]
\( b \quad h(2x - 1) = -2 \)
\[
\therefore 10 - 6x = -2 \quad \{\text{using } a\}
\]
\[
\therefore -6x = -12
\]
\[
\therefore x = 2
\]

6  \( a \quad (f \circ g)(x) = f(g(x))
\quad = f(\sqrt{x})
\quad = 1 - 2\sqrt{x}
\]
\( b \quad (g \circ f)(x) = g(f(x))
\quad = g(1 - 2x)
\quad = \sqrt{1 - 2x}
\]

7  \( f(x) = ax^2 + bx + c, \) where \( f(0) = 5, \ f(-2) = 21, \) and \( f(3) = -4 \)
When \( f(0) = 5, \)
\[
5 = a(0)^2 + b(0) + c
\]
\[
5 = c
\]
\[
\therefore c = 5 \quad \text{... (1)}
\]
When \( f(-2) = 21, \)
\[
21 = a(-2)^2 + b(-2) + c
\]
\[
21 = 4a - 2b + c
\]
\[
\therefore 4a - 2b + 5 \quad \text{... (1)}
\]
\[
\therefore 4a - 2b = 16
\]
\[
2a - b = 8 \quad \text{and so } \ b = 2a - 8 \quad \text{... (2)}
\]
When \( f(3) = -4, \)
\[
-4 = a(3)^2 + b(3) + c
\]
\[
-4 = 9a + 3b + c
\]
\[
\therefore -4 = 9a + 3b + 5 \quad \{\text{using (1)}\}
\]
\[
\therefore -4 = 9a + 3(2a - 8) + 5 \quad \{\text{using (2)}\}
\]
\[
\therefore -4 = 9a + 6a - 24 + 5
\]
\[
\therefore 15 = 15a \quad \text{and so } \ a = 1
\]

Now, substituting \( a = 1 \) into (2) gives \( b = 2(1) - 8 = -6 \)
So, \( a = 1, \ b = -6, \ c = 5.\)

8  \( a \quad f(x) = y = 7 - 4x \)
\[
\therefore f^{-1} \text{ is } \ x = 7 - 4y
\]
\[
\therefore y = \frac{7 - x}{4}
\]
So, \( f^{-1}(x) = \frac{7 - x}{4}.\)

9  \( a \quad f \text{ is } \ y = 7 - 4x \)
\[
\therefore f^{-1} \text{ is } \ x = 7 - 4y
\]
\[
\therefore y = \frac{7 - x}{4}
\]
So, \( f^{-1}(x) = \frac{7 - x}{4}.\)

\( b \quad f \text{ is } \ y = \frac{3 + 2x}{5} \)
\[
\therefore f^{-1} \text{ is } \ x = \frac{3 + 2y}{5}
\]
\[
\therefore 5x = 3 + 2y
\]
\[
\therefore y = \frac{5x - 3}{2}
\]
So, \( f^{-1}(x) = \frac{5x - 3}{2}.\)
10 \[ f \text{ is } y = 5x - 2 \]
\[ \therefore f^{-1} \text{ is } x = 5y - 2 \]
\[ \therefore y = \frac{x + 2}{5} \]
\[ \therefore f^{-1}(x) = \frac{x + 2}{5} \]
\[ h \text{ is } y = \frac{3x}{4} \]
\[ \therefore h^{-1} \text{ is } x = \frac{3y}{4} \]
\[ \therefore y = \frac{4x}{3} \]
\[ \therefore h^{-1}(x) = \frac{4x}{3} \]

Now \( (f^{-1} \circ h^{-1})(x) = f^{-1}(h^{-1}(x)) \)
\[ = f^{-1}\left(\frac{4x}{3}\right) \]
\[ = \frac{4x}{3} + 2 \]
\[ = \frac{4x + 6}{3} \]
\[ = \frac{15x - 6}{4} \]

So, \( y = \frac{15x - 6}{4} \)
\[ \therefore (h \circ f)^{-1}(x) \text{ is } x = \frac{15y - 6}{4} \]
\[ \therefore 4x = 15y - 6 \]
\[ \therefore y = \frac{4x + 6}{15} \]
\[ \therefore (h \circ f)^{-1}(x) = \frac{4x + 6}{15} \]

Hence, \( (f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) \) as required.

11 \[ f \text{ is } y = 2x + 11 \]
\[ g(x) = x^2 \]
\[ \text{so } f^{-1}(x) \text{ is } x = 2y + 11 \]
\[ (g \circ f^{-1})(x) = g(f^{-1}(x)) \]
\[ \therefore y = \frac{x - 11}{2} \]
\[ \therefore f^{-1}(x) = \frac{x - 11}{2} \]
\[ \therefore (g \circ f^{-1})(3) = \left(\frac{3 - 11}{2}\right)^2 \]
\[ = 16 \]

12 The domain is \( \{x \mid x \neq 4\} \), so \( x = 4 \) is a vertical asymptote.
The range is \( \{y \mid y \neq -1\} \), so \( y = -1 \) is a horizontal asymptote.

We now consider the behaviour of the function near the asymptotes, using the sign diagram

As \( x \to -\infty \), \( y \to 1 \)
As \( x \to -\infty \), \( y \to -1 \)

Note that we cannot tell whether the function tends to \(-1\) from above or below.

As \( x \to 4^- \), \( y \to \infty \)
As \( x \to 4^+ \), \( y \to -\infty \)

So, the function could be:
(Note: There may be other answers.)