Arc length and sector area

The diagram shows two circles with centres at the points A and B and radii $2r$ and $r$, respectively. The point B lies on the circle with centre A. The circles intersect at the points C and D.

Let $\alpha$ be the measure of the angle CAD and $\theta$ be the measure of the angle CBD in radians.

1a. Find an expression for the shaded area in terms of $\alpha$, $\theta$ and $r$. [3 marks]

Markscheme

$A = 2(\alpha - \sin \alpha)r^2 + \frac{1}{2}(\theta - \sin \theta)r^2$  \textbf{M1A1A1}

\textbf{Note:} Award \textbf{M1A1A1} for alternative correct expressions \textit{eg.}

$A = 4 \left( \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) r^2 + \frac{1}{2} \theta r^2$.

[3 marks]

1b. Show that $\alpha = 4 \arcsin \frac{1}{4}$. [2 marks]
Markscheme

METHOD 1
consider for example triangle ADM where M is the midpoint of BD \( M1 \)
\[
\sin \frac{\alpha}{4} = \frac{1}{4} \quad A1
\]
\[
\frac{\alpha}{4} = \arcsin \frac{1}{4} \quad A1
\]
\[
\alpha = 4 \arcsin \frac{1}{4} \quad AG
\]

METHOD 2
attempting to use the cosine rule (to obtain \( 1 - \cos\frac{\alpha}{2} = \frac{1}{8} \)) \( M1 \)
\[
\sin \frac{\alpha}{4} = \frac{1}{4} \quad A1
\]
\[
\frac{\alpha}{4} = \arcsin \frac{1}{4} \quad A1
\]
\[
\alpha = 4 \arcsin \frac{1}{4} \quad AG
\]

METHOD 3
\[
\sin \left( \frac{\pi}{2} - \frac{\alpha}{4} \right) = 2 \sin \frac{\alpha}{2} \quad M1
\]
\[
\cos \frac{\alpha}{4} = 4 \sin \frac{\alpha}{4} \cos \frac{\alpha}{4}
\]

Note: Award \( M1 \) either for use of the double angle formula or the conversion from sine to cosine.

\[
\frac{1}{4} = \sin \frac{\alpha}{4} \quad A1
\]
\[
\frac{\alpha}{4} = \arcsin \frac{1}{4} \quad A1
\]
\[
\alpha = 4 \arcsin \frac{1}{4} \quad AG
\]

\( [2 \text{ marks}] \)

1c. Hence find the value of \( r \) given that the shaded area is equal to 4. \( [3 \text{ marks}] \)
2. The diagram below shows a fenced triangular enclosure in the middle of a large grassy field. The points A and C are 3 m apart. A goat $G$ is tied by a 5 m length of rope at point A on the outside edge of the enclosure.

Given that the corner of the enclosure at C forms an angle of $\theta$ radians and the area of field that can be reached by the goat is 44 m², find the value of $\theta$. 

\[ \theta = \pi - \frac{\alpha}{2} \left( = \pi - 2 \arcsin \frac{1}{4} = 2 \arcsin \frac{1}{4} = 2.6362 \ldots \right) \]

\[ r = 1.69 \quad A1 \]

\[ A1 \]

\[ [3 \text{ marks}] \]

\[ [6 \text{ marks}] \]
Markscheme

attempting to use the area of sector formula (including for a semicircle) \( M1 \)

semi-circle \( \frac{1}{2} \pi \times 5^2 = \frac{25\pi}{2} = 39.26990817 \ldots \) \( (A1) \)

angle in smaller sector is \( \pi - \theta \) \( (A1) \)

area of sector = \( \frac{1}{2} \times 2^2 \times (\pi - \theta) \) \( (A1) \)

attempt to total a sum of areas of regions to 44 \( (M1) \)

\[ 2(\pi - \theta) = 44 - 39.26990817 \ldots \]

\[ \theta = 0.777 \left( = \frac{29\pi}{4} - 22 \right) \] \( A1 \)

Note: Award all marks except the final \( A1 \) for correct working in degrees.

Note: Attempt to solve with goat inside triangle should lead to nonsense answer and so should only receive a maximum of the two \( M \) marks.

[6 marks]

3. The following diagram shows a sector of a circle where \( \angle AOB = x \) radians \( [4 \text{ marks}] \) and the length of the arc \( AB = \frac{2}{x} \) cm.

Given that the area of the sector is 16 cm\(^2\), find the length of the arc \( AB \).
Farmer Bill owns a rectangular field, 10 m by 4 m. Bill attaches a rope to a wooden post at one corner of his field, and attaches the other end to his goat Gruff.

4a. Given that the rope is 5 m long, calculate the percentage of Bill’s field that Gruff is able to graze. Give your answer correct to the nearest integer.
**Markscheme**

**EITHER**

area of triangle = $\frac{1}{2} \times 3 \times 4 \ (= 6) \ \ A1$

area of sector = $\frac{1}{2} \arcsin \left( \frac{4}{5} \right) \times 5^2 \ (= 11.5911 \ldots) \ \ A1$

**OR**

$$\int_{0}^{4} \sqrt{25 - x^2} \, dx \quad M1A1$$

**THEN**

total area = 17.5911\ldots \ m^2 \ (A1)

percentage = $\frac{17.5911\ldots}{40} \times 100 = 44\% \ \ A1$

[4 marks]

4b. Bill replaces Gruff’s rope with another, this time of length $a$, $4 < a < 10$, [4 marks]
so that Gruff can now graze exactly one half of Bill’s field.

Show that $a$ satisfies the equation

$$a^2 \arcsin \left( \frac{4}{a} \right) + 4\sqrt{a^2 - 16} = 40.$$
Markscheme

METHOD 1

area of triangle = \( \frac{1}{2} \times 4 \times \sqrt{a^2 - 16} \) \( A1 \)

\[ \theta = \arcsin\left(\frac{4}{a}\right) \quad (A1) \]

area of sector = \( \frac{1}{2} r^2 \theta = \frac{1}{2} a^2 \arcsin\left(\frac{4}{a}\right) \) \( A1 \)

therefore total area = \( 2\sqrt{a^2 - 16} + \frac{1}{2} a^2 \arcsin\left(\frac{4}{a}\right) = 20 \) \( A1 \)

rearrange to give: \( a^2 \arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40 \) \( AG \)

METHOD 2

\[
\int_{0}^{4} \sqrt{a^2 - x^2} \, dx = 20 \quad M1
\]

use substitution \( x = a \sin \theta, \quad \frac{dx}{d\theta} = a \cos \theta \)

\[
\int_{0}^{\arcsin\left(\frac{4}{a}\right)} a^2 \cos^2 \theta \, d\theta = 20
\]

\[
\frac{a^2}{2} \int_{0}^{\arcsin\left(\frac{4}{a}\right)} (\cos 2\theta + 1) \, d\theta = 20 \quad M1
\]

\[
a^2 \left[ \left(\frac{\sin 2\theta}{2} + \theta \right) \right]_{0}^{\arcsin\left(\frac{4}{a}\right)} = 40 \quad A1
\]

\[
a^2 [(\sin \theta \cos \theta + \theta)]_{0}^{\arcsin\left(\frac{4}{a}\right)} = 40
\]

\[
a^2 \arcsin\left(\frac{4}{a}\right) + a^2 \left(\frac{4}{a}\right) \sqrt{\left(1 - \left(\frac{4}{a}\right)^2\right)} = 40 \quad A1
\]

\[
a^2 \arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40 \quad AG
\]

[4 marks]

4c. Find the value of \( a. \) \( [2 \text{ marks}] \)
5. The radius of the circle with centre C is 7 cm and the radius of the circle with centre D is 5 cm. If the length of the chord [AB] is 9 cm, find the area of the shaded region enclosed by the two arcs AB.

\[ \text{area of shaded region} = \text{area of \( \triangle ABC \)} - \text{area of sector} \]

\[ \text{area of \( \triangle ABC \)} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 9 \times 7 = 31.5 \text{ cm}^2 \]

\[ \text{area of sector} = \frac{1}{2} \times \text{radius}^2 \times \theta \]

\[ \theta = \frac{180 \times \text{chord length}}{2 \times \text{radius}^2} = \frac{180 \times 9}{2 \times 7^2} = \frac{1620}{98} \approx 16.47 \text{ degrees} \]

\[ \text{area of sector} = \frac{16.47}{360} \times \pi \times 7^2 = \frac{16.47}{360} \times 49\pi \approx 27.6 \text{ cm}^2 \]

\[ \text{area of shaded region} = 31.5 - 27.6 = 3.9 \text{ cm}^2 \]
\[
\alpha = 2 \arcsin \left( \frac{4.5}{7} \right) \Rightarrow \alpha = 1.396... = 80.010^\circ \ldots \quad \text{M1(A1)}
\]
\[
\beta = 2 \arcsin \left( \frac{4.5}{5} \right) \Rightarrow \beta = 2.239... = 128.31^\circ \ldots \quad \text{(A1)}
\]

Note: Allow use of cosine rule.

\[
\text{area } P = \frac{1}{2} \times 7^2 \times (\alpha - \sin \alpha) = 10.08... \quad \text{M1(A1)}
\]
\[
\text{area } Q = \frac{1}{2} \times 5^2 \times (\beta - \sin \beta) = 18.18... \quad \text{(A1)}
\]

Note: The \textbf{M1} is for an attempt at area of sector minus area of triangle.

Note: The use of degrees correctly converted is acceptable.

\[
\text{area} = 28.3 \text{ (cm}^2\text{)} \quad \text{A1}
\]

[7 marks]
The diagram below shows two concentric circles with centre O and radii 2 cm and 4 cm.

The points P and Q lie on the larger circle and PÒQ = x, where

\[0 < x < \frac{\pi}{2} \).

6. (a) Show that the area of the shaded region is \(8 \sin x - 2x\). [7 marks]

(b) Find the maximum area of the shaded region.

**Markscheme**

(a) shaded area area of triangle area of sector, i.e. \((M1)\)

\[
\left( \frac{1}{2} \times 4^2 \sin x \right) - \left( \frac{1}{2} \cdot 2^2 x \right) = 8 \sin x - 2x
\]

\(A1A1AG\)

(b) **EITHER**

any method from GDC gaining \(x \approx 1.32\) \((M1)(A1)\)

maximum value for given domain is 5.11 \(A2\)

**OR**

\[
\frac{dA}{dx} = 8 \cos x - 2 \quad A1
\]

set \(\frac{dA}{dx} = 0\), hence \(8 \cos x - 2 = 0\) \(M1\)

\(\cos x = \frac{1}{4} \Rightarrow x \approx 1.32\) \(A1\)

hence \(A_{\text{max}} = 5.11\) \(A1\)

[7 marks]
7. The diagram below shows two straight lines intersecting at O and two circles, each with centre O. The outer circle has radius $R$ and the inner circle has radius $r$.

Consider the shaded regions with areas $A$ and $B$. Given that $A : B = 2 : 1$, find the exact value of the ratio $R : r$.

**Markscheme**

\[
A = \frac{\theta}{2}(R^2 - r^2) \quad A1 \\
B = \frac{\theta}{2}r^2 \quad A1 \\
\text{from } A : B = 2 : 1, \text{ we have } R^2 - r^2 = 2r^2 \quad M1 \\
R = \sqrt{3}r \quad (A1) \\
\text{hence exact value of the ratio } R : r \text{ is } \sqrt{3} : 1 \quad A1 \quad N0
\]

[5 marks]

8. A circular disc is cut into twelve sectors whose areas are in an arithmetic sequence.

The angle of the largest sector is twice the angle of the smallest sector.

Find the size of the angle of the smallest sector.
Markscheme

METHOD 1
If the areas are in arithmetic sequence, then so are the angles. \( \text{M1} \)
\[ S_n = \frac{n}{2} (a + l) \Rightarrow \frac{12}{2} (\theta + 2\theta) = 18\theta \quad \text{M1A1} \]
\[ \Rightarrow 18\theta = 2\pi \quad \text{(A1)} \]
\[ \theta = \frac{\pi}{9} \quad \text{(accept 20°)} \quad \text{A1} \]

[5 marks]

METHOD 2
\[ a_{12} = 2a_1 \quad \text{(M1)} \]
\[ \frac{12}{2} (a_1 + 2a_1) = \pi r^2 \quad \text{M1A1} \]
\[ 3a_1 = \frac{\pi r^2}{6} \]
\[ \frac{3}{2} r^2 \theta = \frac{\pi r^2}{6} \quad \text{(A1)} \]
\[ \theta = \frac{2\pi}{18} = \frac{\pi}{9} \quad \text{(accept 20°)} \quad \text{A1} \]

[5 marks]

METHOD 3
Let smallest angle = \( a \), common difference = \( d \)
\[ a + 11d = 2a \quad \text{(M1)} \]
\[ a = 11d \quad \text{A1} \]
\[ S_n = \frac{12}{2} (2a + 11d) = 2\pi \quad \text{M1} \]
\[ 6(2a + a) = 2\pi \quad \text{(A1)} \]
\[ 18a = 2\pi \]
\[ a = \frac{\pi}{9} \quad \text{(accept 20°)} \quad \text{A1} \]

[5 marks]