3 Dividing up space: coordinate geometry, Voronoi diagrams, vectors, lines

Skills check

1 \( d = \sqrt{(1-5)^2 + (2-4)^2} = \sqrt{16 + 4} = \sqrt{20} = 4.47 \)

2 \( \left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3,3) \)

3 a \( \cos(40) = \frac{x}{10} \). Hence, \( x = 7.66 \).

\[ \frac{y}{10} = \sin(40) \]. Hence, \( y = 6.43 \).

b \( \tan(x) = \frac{6}{9} \). Hence, \( x = \arctan\left(\frac{6}{9}\right) = 33.7^\circ \).

4 a i \( \frac{x}{50} = \sin(60) \). Hence, \( x = 43.3 \) km east of A.

ii \( \frac{y}{50} = \cos(60) \). Hence, \( y = 25.0 \) km north of A.

b \( x = 270 - (180 - 90 - 60) = 270 - 30 = 240^\circ \).

Exercise 3A

1 a i \( d = \sqrt{((2-1)^2 + (1-4)^2)} = \sqrt{1 + 25} = \sqrt{26} = 5.10 \)

ii \( d = \sqrt{(2-0)^2 + (4-3)^2 + (-3+2)^2} = \sqrt{4 + 1 + 1} = \sqrt{6} = 2.45 \)

b i \( m = \left(\frac{2+1}{2}, \frac{(1-4)}{2}\right) = (1.5, -1.5) \).

ii \( m = \left(\frac{2+0}{2}, \frac{(4+3)}{2}, \frac{-3-2}{2}\right) = (1,3.5,-2.5) \).

2 a i \( AB = \sqrt{(21-(-3))^2 + ((-13)-14)^2} = \sqrt{576 + 729} = \sqrt{1305} = 36.1 \)

ii \( AB = \sqrt{(-17-(-2))^2 + (11-8)^2 + (0-(-12))^2} = \sqrt{378} = 19.4 \)

b i \( m = \left(\frac{21-3}{2}, \frac{14-13}{2}\right) = (9,0.5) \)

ii \( m = \left(\frac{-17-2}{2}, \frac{11+8}{2}, \frac{0-12}{2}\right) = (-9.5,9.5,-6) \).

3 a \( d = \sqrt{(26-20)^2 + (31-25)^2 + (12-11)^2} = \sqrt{36 + 36 + 1} = \sqrt{73} = 8.54 \) km

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\[d_1 = \sqrt{20^2 + 25^2 + 11^2} = \sqrt{1146} = 33.9 < 40. \text{ The first aircraft can be detected.}\]
\[d_2 = \sqrt{26^2 + 31^2 + 12^2} = \sqrt{1781} = 42.2 > 40. \text{ The second aircraft cannot be detected.}\]

4 \ a \ A = (250, 0, 0), B = (250, 400, 0), C = (0, 400, 60), D = (0, 0, 60).
\[M = \left( \frac{250 + 0}{2}, \frac{400 + 400}{2}, \frac{0 + 60}{2} \right) = (125, 400, 30).\]
\[l = \sqrt{(250 - 125)^2 + (0 - 400)^2 + (0 - 30)^2} + \sqrt{(125 - 0)^2 + (400 - 0)^2 + (30 - 60)^2}\]
\[= \sqrt{176525} + \sqrt{176525} = 2\sqrt{176525} = 840 \text{ m}\]

5 \ a \ AB = \sqrt{(340 - 97)^2 + (77 - (-139))^2 + (21 - 21)^2} = \sqrt{59049 + 46656 + 0} = \sqrt{105705}.

If the length of the base is \(x\) then \(2x^2 = AB\)

length of a side of the base \(\frac{AB}{\sqrt{2}} = \frac{105705}{2}\).

Area of the base \(\left( \frac{AB}{\sqrt{2}} \right)^2 = \frac{105705}{2} = 52900 \text{ m}^2\).

b volume \(\frac{\text{area of base} \times \text{height}}{3} = \frac{52900 \times 138}{3} = 2430000 \text{ m}^3\)

c midpoint of \(AB = \left( \frac{340 + 97}{2}, \frac{77 - (-139)}{2}, \frac{-21 - 21}{2} \right) = (218.5, -31, -21).\)

Vertex is 138m above this, so at \((218.5, -31, 117)\).

d \(d = \sqrt{(340 - 218.5)^2 + (-139 - (-31))^2 + (-21 - 117)^2} = 213 \text{ m}\)

Exercise 3B

1 \ a \ \(x = 2\) \quad \(b \ y = 6\) \quad \(c \ (2,6)\)
2 \ a \ \(y = 3x + 5\) \quad \(b \ y = -2x + 0.4\) \quad \(c \ y = 4.5x + 5\)

\[d \ y = 2x + c.\]
\[5 = 2 \times 3 + c.\]
\[c = -1.\]
\[y = 2x - 1.\]

3 \ a \ \(m = \frac{-3 - 5}{5 - 3} = -4. \quad 5 = 3 \times -4 + c. \quad c = 17. \quad y = -4x + 17.\)

\[b \ m = \frac{4 - (-1)}{3 - 2} = 5. \quad 4 = 5 \times 3 + c. \quad c = -11. \quad y = 5x - 11.\]

\[c \ m = \frac{4 - 2}{3 - 3} = \frac{2}{6} = \frac{1}{3}. \quad 4 = \frac{1}{3} \times 3 + c. \quad c = 3. \quad y = \frac{1}{3}x + 3.\]
4. a. \( m = \frac{8 - (-4)}{6 - 12} = \frac{12}{6} = -2 \). \( 8 = -2 \times 6 + c \). \( c = 20 \). \( y = -2x + 20 \).

b. i. \( y = -2 \times 0 + 20 = 20 \). \((0, 20)\).

ii. \( y = -2 \times 22 + 20 = -24 \). \((22, -24)\).

c. \( \sqrt{(0 - 22)^2 + (20 - (-24))^2} = \sqrt{484 + 1936} = \sqrt{2420} = 49.2 \text{ km} \)

Exercise 3C

1. a. i. \( y - 9 = 2(x - 3) \)

ii. \( y = 2x + 3 \)

iii. \( 2x - y + 3 = 0 \)

b. i. \( y - 5 = \frac{1}{2}(x - 6) \)

ii. \( y = \frac{1}{2}x + 2 \)

iii. \( x - 2y + 4 = 0 \)

c. i. \( y + 7 = -\frac{1}{3}(x - 6) \)

ii. \( y = -\frac{1}{3}x - 5 \)

iii. \( x + 3y + 15 = 0 \)

2. a. \( m = \frac{11 - 5}{5 - 2} = \frac{6}{3} = 2 \).

i. \( y - 5 = 2(x - 2) \)

ii. \( y = 2x + 1 \)

iii. \( -2x + y - 1 = 0 \)

b. \( m = \frac{4 - 2}{0 - 2} = -1 \).

i. \( y - 4 = -x \)

ii. \( y = -x + 4 \)

iii. \( x + y - 4 = 0 \)

C. \( m = \frac{9 - 6}{3 - 2} = 3 \).

i. \( y - 6 = 3(x - 2) \)

ii. \( y = 3x \)
iii  $3x - y = 0$

\[ d \quad m = \frac{-8 + 6}{1 + 2} = \frac{-2}{3} \]

i  $y + 6 = \frac{-2}{3}(x + 2)$

ii  $y = \frac{-2}{3}x + \frac{22}{3}$

iii  $2x + 3x + 22 = 0$

3  

\[ a \quad 3y = 2x - 7 \Rightarrow y = \frac{2}{3}x - \frac{7}{3}, \text{ Gradient is } \frac{2}{3}. \]

\[ b \quad 7y = -4x + 6 \Rightarrow y = -\frac{4}{7}x + \frac{6}{7}, \text{ Gradient is } -\frac{4}{7}. \]

\[ c \quad by = -ax - d \Rightarrow y = -\frac{a}{b}x - \frac{d}{b}, \text{ Gradient is } -\frac{a}{b}. \]

\[ 4 \quad a \quad m = \frac{5 - 2}{1 - 3} = \frac{3}{2} \Rightarrow y = -\frac{3}{2}x + c \Rightarrow 5 = -\frac{3}{2} \times 1 + c \Rightarrow c = \frac{13}{2}. \]

\[ y = -\frac{3}{2}x + \frac{13}{2}. \]

\[ b \quad 2y = -3x + 13 \Rightarrow 3x + 2y - 13 = 0. \]

\[ c \quad \text{The height of the garden is } y = c = \frac{13}{2} \text{ m.} \]

\[ \text{The width of the garden satisfies } 0 = -3x + 13 \text{ so is } x = \frac{13}{3} \text{ m.} \]

\[ \text{The area of grass he needs to buy is } \frac{1}{2} \left( \frac{13}{2} \times \frac{13}{3} \right) = \frac{169}{12} = 14.1 \text{ m}^2. \]

Exercise 3D

1  

\[ a \quad i \quad 2x + y = 8 \Rightarrow 4x + 2y = 16. \text{ Add this to the other equation to get} \]

\[ 7x = 49 \Rightarrow x = 7 \Rightarrow y = 8 - 14 = -6. \]

Solution is $x = 7, y = 6$.

\[ ii \quad 2x + 10y = 3 \Rightarrow 6x + 30y = 9. \quad 3x + 15y = 4.5 \Rightarrow 6x + 30y = 9. \]

Two lines are the same.

\[ iii \quad \text{Substitute the first equation into the second to get } 4x - 2x - 1 = 5 \Rightarrow 2x = 6 \Rightarrow x = 3. \]

So $y = 2 \times 3 + 1 = 7$.

Solution is $x = 3, y = 7$.

\[ iv \quad \text{Substitute the second equation into the first: } 2x - 11 = 3x - 12 \Rightarrow x = 1 \Rightarrow y = 2 \times 1 - 9 \]

Solution is $x = 1, y = -7$. 
\[ \textbf{b i} \quad x + 3y = 1 \Rightarrow 5x + 15y = 5. \]

Subtract this from second equation to get \( y = 3. \)

\[ x = 1 - 3y \Rightarrow x = -8. \]

Solution is \( x = -8, y = 3. \)

\[ \textbf{ii} \quad 3x + 2y = 4 \Rightarrow 6x + 4y = 8. \quad \text{Add this to second equation to get} \]

\[ 11x = 28.6 \Rightarrow x = 2.6. \]

\[ 3 \times 2.6 + 2y = 4 \Rightarrow y = -1.9. \]

Solution is \( x = 2.6, y = -1.9. \)

\[ \textbf{2 a} \quad y = \frac{1}{2} \times 50 - 100 = 25 - 100 = -75. \quad \text{So Bernard is on the road.} \]

\[ \textbf{b} \quad \text{Subtract the first equation from the second:} \]

\[ \frac{3}{2} x - 510 = 0 \Rightarrow x = 340. \quad y = 410 - 340 = 70. \]

Roads intersect at \( (340,70). \)

\[ \textbf{c i} \quad \text{The distance Alison has to walk is} \quad \sqrt{(340 - 0)^2 + (70 - 410)^2} = 480.8\ldots \text{m}. \]

The distance Bernard has to walk is \( \sqrt{(340 - 50)^2 + (70 - (-75))^2} = 324.2\ldots \text{m}. \)

So Bernard arrives first.

\[ \text{ii} \quad \text{Alison needs to travel an extra} \quad 480.8\ldots - 324.2\ldots \approx 156.6 \text{m, which will take} \]

\[ 156.6 \approx 0.03915\ldots \text{ hours or} \quad 2.349 \text{ minutes or} \quad 141 \text{ seconds.} \]

\[ \textbf{3 a} \quad \frac{5}{20} \times 100 = 25\%. \]

\[ \textbf{b} \quad \text{The way down is steeper.} \]

\[ \textbf{c i} \quad m = 0.1. \quad c = 0. \]

\[ y = 0.1x. \]

\[ \textbf{ii} \quad m = -0.15. \quad \text{Passes through the point} \quad (2.45,0). \]

\[ 0 = -0.15 \times 2.45 + c. \quad c = 0.3675. \]

\[ y = -0.15x + 0.3675 \]

\[ \textbf{d} \quad \text{Substitute the first equation into the second:} \]

\[ 0.1x = -0.15x + 0.3675 \Rightarrow x = 1.47. \]

Height of the hill is \( y = 0.1 \times 1.47 = 0.147 \text{ km or} \quad 147 \text{m}. \]
e The distance up is \( \sqrt{(1.47)^2 + 0.147^2} = 1.477 \text{ km.} \)

The distance down is \( \sqrt{(2.45-1.47)^2 + 0.147^2} = 0.9910 \text{ km.} \)

So the total distance is 2.47 \text{ km.}

4 a Pick any point on the straight line (except the one on the x axis). The straight line defines a right-angled triangle, with the chosen point as one of the vertices. The straight line is the hypotenuse, the x-axis contains the adjacent side with respect to \( \alpha \) and the line vertically down from the chosen point forms the opposite side. Then the gradient of the line is  
\[
\frac{\text{change of y coordinate}}{\text{length of opposite side}} = \frac{\text{change in x coordinate}}{\text{length of adjacent side}} = \tan \alpha.
\]

b \( m = \tan 4 = 0.069927 \)

\[
580 = 0.069927 \times 7500 + c. \quad c = 55.549.
\]

y = 0.0699x + 55.5

c \( y = c = 55.5 \text{ m.} \)

d \( 0 = 0.0699x + 55.5 \Rightarrow x = -794. \)

The aircraft lands \( 794 - 700 = 94 \text{ m} \) from the start of the runway.

**Exercise 3E**

1 a \( \frac{1}{2} \)  

b -3  

c -2  

d \( \frac{7}{6} \)

2 a Gradient of \( bx - ay = d_1 \) is \( m_1 = \frac{b}{a} \). Gradient of \( ac + by = d_2 \) is \( m_2 = \frac{a}{b} \).

\[
m_1m_2 = \frac{b}{a} \cdot \left(-\frac{a}{b}\right) = -1.
\]

b \( x + 2y - 10 = 0 \)

So, \( b = 1, a = -2, d_1 = 10 \)

Perpendicular line:

\[
ax + by = d_2
\]

\[
-2x + y = d_2
\]

\[
-2 \times 2 + 5 = d_2
\]

\[
d_2 = 1
\]

Hence, \( -2x + y = 1 \) or \( 2x - y + 1 = 0 \)

c \( 3x - 2y = 7 \)

So, \( b = 3, a = 2, d_1 = 7 \)

Perpendicular line:
\[ ax + by = d_2 \]
\[ 2x + 3y = d_2 \]
\[ 2 \times 6 + 3 \times 5 = d_2 \]
\[ d_2 = 27 \]

Hence, \(2x + 3y = 27\) or \(2x + 3y - 27 = 0\)

3 a  If you have any other point, B, on the line then this point, A and the point of intersection between the line and its perpendicular line through A form a right-angled triangle. But, the hypotenuse of this triangle is the line joining A and B and so is longer than the length of the perpendicular line.

\[ 4x + 3y = d_1, \quad d_1 = 4 \times 5 + 3 \times (-7) = -1. \quad 4x + 3y + 1 = 0. \]

\[ 3x - 4y + 7 = 0 \Rightarrow 9x - 12y + 21 = 0. \quad 4x + 3y + 1 = 0 \Rightarrow 16x + 12y + 4 = 0 \]

\[ 25x + 25 = 0 \Rightarrow x = -1. \quad 3 \times -1 - 4 \times y + 7 = 0 \Rightarrow y = 1. \]

Point of intersection is \((-1, 1)\).

\[ \text{Shortest distance of A from } l \text{ is } \sqrt{(5 - (-1))^2 + (-7 - 1)^2} = \sqrt{36 + 64} = 10 \]

4 a  Perpendicular line is \(5x - 3y = d_1, \quad d_1 = 5 \times 2 - 3 \times 4 = -2. \quad 5x - 3y + 2 = 0. \)

Finding the point of intersection:
\[ 5x - 3y + 2 = 0 \Rightarrow 25x - 15y + 10 = 0 \]
\[ 3x + 5y + 8 = 0 \Rightarrow 9x + 15y + 24 = 0. \]

Adding these two equations:
\[ 34x = -34 \Rightarrow x = -1. \]
\[ 3 \times -1 + 5y + 8 = 0 \Rightarrow y = -1. \]

Point of intersection is \((-1, -1)\).

\[ \text{Shortest distance is } \sqrt{(2 + 1)^2 + (4 + 1)^2} = \sqrt{9 + 25} = \sqrt{34}. \]

b  Perpendicular line: \[ y = -\frac{1}{3}x + c. \]
\[ -1 = -\frac{1}{3} \times 5 + c. \]
\[ c = \frac{2}{3}. \]
\[ y = -\frac{1}{3}x + \frac{2}{3} \]

Finding the point of intersection: Take the equation of the perpendicular line away from the equation of the original line
\[ 0 = \frac{10}{3} x - \frac{8}{3} \Rightarrow x = \frac{4}{5}. \]
\[ y = 3 \times \frac{4}{5} - 2 \times \frac{2}{5}. \]
\[ \left(\frac{4}{5}, \frac{2}{5}\right). \]
Shortest distance is \( \sqrt{(5 - \frac{4}{5})^2 + (-1 - \frac{2}{5})^2} = \sqrt{19.6} = 4.43 \).

5 The gradient of AB is \( \frac{18 - 16}{12 - 17} = \frac{2}{5} \).

The midpoint of this line is \( \left(\frac{12 + 17}{2}, \frac{18 + 16}{2}\right) = (14.5, 17) \).

Perpendicular bisector is
\[
y = 2.5x + c \quad \text{and} \quad 17 = 2.5 \times 14.5 + c \quad \Rightarrow \quad c = -19.25.
\]

Line to the northeast passing through A is \( y = x + c \) \( \quad \Rightarrow \quad 18 = 12 + c \quad \Rightarrow \quad c = 6 \). \( y = x + 6 \).

Point of intersection: \( 2.5x - 19.25 = x + 6 \Rightarrow x = \frac{101}{6} = 16.83 \).

\( y = x + 6 = 22.83 \).

The ship is at \( (16.8, 22.8) \) when it is north east of A.

6 a We need to minimise AS + SB. The minimum value is when the three points are collinear. Hence, S lines on the line between A and B, and also on \( y = x + 10 \). (Note that S is not necessarily between A and B.)

The gradient of the line between the two towns is \( m = \frac{24 - 16}{17 - 1} = \frac{8}{16} = \frac{1}{2} \).

So the equation of the line between the two towns is \( y - 16 = \frac{1}{2}(x - 1) \).

To find the point of intersection, substitute the track equation in the equation above
\[
x + 10 - 16 = \frac{1}{2}(x - 1) \Rightarrow x = 11.
\]
\[
y = x + 10 = 21.
\]
Station should be built at \( (11, 21) \).

Total distance is \( \sqrt{(17 - 1)^2 + (24 - 16)^2} \approx 17.9 \) km.

b To be the same distance from each town the station must lie on the perpendicular bisector of \([AB]\). Finding the perpendicular bisector: Midpoint of \([AB]\) is \( \left(\frac{1 + 17}{2}, \frac{16 + 24}{2}\right) = (9, 20) \).

The gradient of the perpendicular bisector is \( \frac{2}{1} = -2 \).

The equation of the perpendicular bisector is
\[
y - 20 = -2(x - 9).
\]
Point of intersection: Substitute the equation of the rail track into the equation above
\[
x + 10 - 20 = -2(x - 9) \Rightarrow 3x = 28 \Rightarrow x = 9 \frac{1}{3}.
\]
\[ y = 9\frac{1}{3} + 10 = 19\frac{1}{3}. \]

Station should be built at \( \left( \frac{9}{3}, 19\frac{1}{3} \right) \)

Total distance is \( 2 \times \sqrt{\left( \frac{9}{3} - 1 \right)^2 + \left( 19\frac{1}{3} - 16 \right)^2} \approx 18.0 \) km.

**Exercise 3F**

1. **a** Finding the perpendicular bisectors:
   
   \[ m_1 = \frac{2 - 1}{3 - 1} = -\frac{1}{2}. \]

   Midpoint of AC is \( \left( \frac{2 + 1}{2}, \frac{3 + 1}{2} \right) = (1.5, 2). \)

   \[ y - 2 = -\frac{1}{2}(x - 3). \]

   \[ m_2 = \frac{2 - 3}{3 - 1} = -\frac{1}{2}. \]

   Midpoint of BC is \( \left( \frac{2 + 3}{2}, \frac{3 + 1}{2} \right) = (2.5, 2). \)

   \[ y - 2 = \frac{1}{2}(x - 5). \]

1. **b** Finding the perpendicular bisectors:
   
   \[ m_1 = \frac{(5 - 1)}{5 - 1} = -1. \]

   Midpoint of AC is \( \left( \frac{5 + 1}{2}, \frac{5 + 1}{2} \right) = (3, 3). \)

   \[ y - 3 = -(x - 3). \]

   \[ m_2 = \frac{3 - 1}{5 - 1} = -\frac{1}{2}. \]

   Midpoint of AD is \( \left( \frac{3 + 1}{2}, \frac{5 + 1}{2} \right) = (2, 3). \)
\[ y - 3 = \frac{1}{2}(x - 2). \]

Perpendicular bisector between A and B is \( y = 3. \)

\[ m_3 = \frac{3 - 5}{5 - 1} = \frac{1}{2}. \]

Midpoint of BD is \( \left( \frac{3 + 5}{2}, \frac{5 + 1}{2} \right) = (4, 3). \)

\[ y - 3 = \frac{1}{2}(x - 4). \]

Perpendicular bisector of DC is \( x = 4 \)

2 a

b The reading at point (1,4) would be 21°C.

3 a Gradient between E and D is: \( m = \frac{3 - 6}{6 - 5} = -3. \)

Midpoint is \( \left( \frac{6 + 5}{2}, \frac{3 + 6}{2} \right) = (5.5, 4.5). \)

Perpendicular bisector: \( y - 4.5 = \frac{1}{3}(x - 5.5) \) or \( x - 3y = 8 \)

b Perpendicular bisector between A and E is \( x = 4. \)
Gradient between B and E is \( m = \frac{0 - 4}{8 - 6} = -2 \).

Midpoint is \( \left( \frac{8 + 6}{2}, \frac{0 + 4}{2} \right) = (7,2) \)

Perpendicular bisector: \( y - 2 = \frac{1}{2}(x - 7) \) or \( x - 2y = 3 \)

Exercise 3G

1 a This point will be a vertex in a Voronoi diagram, as otherwise the point will be in a cell, and can therefore be moved further away from the corresponding school by moving the point to a vertex of that cell. The only vertex of the Voronoi diagram in this example is the meeting point of the three perpendicular bisectors of [AB], [AC] and [BC].

b Perpendicular bisector of [AB]: \( m = \frac{-6 - 1}{4 - 3} = -5 \).

Midpoint is \( \left( \frac{1 + 6}{2}, \frac{3 + 4}{2} \right) = (3.5, 3.5) \).

\( y - 3.5 = -5(x - 3.5) \)

\( y = -5x + 21 \)

Perpendicular bisector of [BC]: \( y = 2.5 \).

c \( 2.5 - 3.5 = -5(x - 3.5) \)

\( -1 = -5x + 17.5 \Rightarrow x = 3.7 \).

New school should be built at \((3.7, 2.5)\).

d This new school is the same distance from the three original schools:

\( d = \sqrt{(3.7 - 1)^2 + (2.5 - 3)^2} \)

\( = 2.75 \text{ km} \)
2 a i  \( m = \frac{40 - 20}{10 - 30} = 1 \).

Midpoint is \( \left( \frac{20 + 40}{2}, \frac{30 + 10}{2} \right) = (30, 20) \).

\( y = x - 10 \).

ii  \( m = \frac{80 - 40}{30 - 10} = -2 \).

Midpoint is \( \left( \frac{80 + 40}{2}, \frac{30 + 10}{2} \right) = (60, 20) \).

\( y = -2x + 140 \).

b

![Graph showing the midpoint calculations]

c i  Area of the fairground that will go to stand C is

\( \frac{1}{2} (60 \times 40) = 1200 \).

Total area of fairground is 5000 .

So proportion that will go to stand C is

\[ \frac{1200}{5000} = 0.24 \]

ii  Area of the fairground that will go to stand A is

\[ (10 \times 50) + (10 \times 40) + \frac{1}{2} (40 \times 40) = 500 + 400 + 800 = 1700. \]

So proportion that will go to stand A is

\[ \frac{17}{50} = 0.34 \]

d i  The stand should be built at the intersection of all three perpendicular bisectors. That is, it is the solution to \( y = x - 10 \) and \( y = -2x + 140 \). Thus,

\[ x - 10 = -2x + 140 \]

\[ 3x = 150 \]

\[ x = 50 \]

\[ y = 40 \]

New stand is at \( (50, 40) \).

ii  \[ d = \sqrt{(50 - 40)^2 + (40 - 10)^2} = 31.6 \]  . The distance from all three other stands is 31.6 m.
3 a \[ m = \frac{3 - 2}{3 - 5} = \frac{1}{2}. \]

midpoint is \( \left( \frac{2 + 3}{2}, \frac{3 + 5}{2} \right) = (2.5, 4). \)

\[ y = \frac{1}{2} x + 2.75. \]

b

![Graph of a line with points plotted on a coordinate plane.]

c

![Graph showing a shaded region with a line and points plotted.]

d i \[ BD: \quad m = \frac{8 - 3}{1 - 3} = \frac{5}{2} \]

midpoint is \( \left( \frac{3 + 8}{2}, \frac{3 + 1}{2} \right) = (5.5, 2). \)

\[ y - 2 = \frac{5}{2} (x - 5.5) \text{ or } y = \frac{5}{2} x - 11.75 \]

CD: \[ y = 3.5 \]

ii B and C are closer to D than they are to A.
iii

\[ y = 3.5 \text{ meets } y = -\frac{5}{3}x + \frac{41}{3} \text{ when } \frac{3.5}{1} = -\frac{5}{3}x + \frac{41}{3} \Rightarrow \frac{41 - 10.5}{5} = 6.1. \]

\[ y = -\frac{5}{3}x + \frac{41}{3} \text{ meets } y = -6x + 35.5 \text{ when } -6x + 35.5 = -\frac{5}{3}x + \frac{41}{3} \Rightarrow 13x = 65.5 \Rightarrow x = 5.038. \]

\[ y = -\frac{5}{3}(5.038) + \frac{41}{3} = 5.27. \]

The vertices are at \((6.1,3.5)\) and \((5.04,5.27)\).

\[ \text{Area C is bounded by } (5.04,5.27), \ (6.1,3.5), \ (10,3.5), \ (10,8) \text{ and a point given by } y = -6x + 35.5 \text{ when } y = 8. \text{ This point is } (4.58,8). \]

Area of upper trapezium = \[ \frac{1}{2}\left[(10 - 4.58) + (10 - 5.04)\right] \times (8 - 5.27) = 14.17 \]

Area of lower trapezium = \[ \frac{1}{2}\left[(10 - 5.04) + (10 - 6.1)\right] \times (5.27 - 3.5) = 7.84 \]

Area of C = 14.2 + 7.8 = 22.0

Total area is 80. So percentage area is \(\frac{22}{80} \times 100 = 28\%\)

\[ \text{Distance from the vertex at } (6.1,3.5) \text{ to the vertex D is } \sqrt{(8 - 6.1)^2 + (1 - 3.5)^2} = 3.14. \]

Distance from the vertex at \((5.038,5.27)\) to \(A\) is \(\sqrt{(5.04 - 2)^2 + (5.27 - 5)^2} = 3.05. \) So the new school should be built at \((6.1,3.5)\).
4 a

b Finding perpendicular bisector of AD: \( m = \frac{8 - 2}{4 - 2} = -3 \).

\[
\left( \frac{8 + 2}{2}, \frac{4 + 2}{2} \right) = (5, 3).
\]

\( y = -3x + 18. \)

This meets \( y = 4 \) when \( 4 = -3x + 18 \Rightarrow x = \frac{14}{3} = 4.67 \)

\((4.67, 4)\).

c i Trapezium: \( \frac{1}{2} \times 4 \times \left( 6 + \frac{14}{3} \right) = \frac{64}{3} = 21.3 \) or 213000 miles\(^2\)

ii Rectangle + trapezium: \( (5 \times 3) + \frac{1}{2} \times 1 \times \left( 5 + \frac{14}{3} \right) = 19.8 \) or 198000 miles\(^2\)

iii Rectangle: \( 3 \times 5 = 15 \) or 150000 miles\(^2\)

iv Not possible to support the other province.

5 a i AB = 6, BC = 3, CD = 4 and DA = \( \sqrt{(2-0)^2 + (5-2)^2} = 3.61 \).

Total distance is \( 6 + 3 + 4 + 3.61 = 16.61 \) or 166 km

ii Distance travelled while closest to A is \( \frac{1}{2} (6 + 3.61) = 4.81 \) or 48.1 km.

So, proportion of journey spent closest to A is \( \frac{48.1}{166} = 0.290 \).

b i This is reasonable because \((4,3.5)\) is a vertex in the Voronoi diagram and so it is an equal distance from three stations. The three stations are B, C and D.

ii \( \frac{2.1 + 2.6 + 2.8}{3} = 2.5 \).

c i Median=3.0. Lower quartile= 2.7. Upper Quartile= 3.1. Interquartile range=0.4

ii Outliers would be below \( 2.7 - 1.5 \times 0.4 = 2.1 \) or above \( 3.1 + 1.5 \times 0.4 = 3.7 \). So the reading of 2.1 is not an outlier because it is on the threshold not below it.
d All the readings, apart from the lowest one, are above the expected value the officer calculated. So, the readings back up the houseowner's claim.

Exercise 3H

1 a \([0,-3] - 3j\)  
   b \([-2, \text{-}2l - 2j]\)  
   c \([2, 2l + j]\)  
   d \([1, -3], i - 3j\)  
   e \([2, 2l]\)

2 a \([-3, 4]\)  
   b \(7l + 4j\)  
   c \([-6, 2]\)  
   d \(5i - 2j\)

3 a \([4, 6]\)  
   b \([6, 9]\)

   c To multiply a vector by a scalar, you multiply each component of the vector by that scalar.

   d \([3i + 4j = 3(1) + 4(0)] = [3(0) + 4(1)] = [3, 4]\).

4 a i \(CD = \frac{1}{2} = AB\)
   ii \(FE = \frac{1}{2} = -AB\)
   iii \(GH = \frac{2}{4} = 2AB\)
   iv \(\overrightarrow{IJ} = \left[-0.5, \frac{1}{2}\right] = \frac{1}{2} \overrightarrow{AB}\)

   b Parallel vectors are scalar multiples of each other.

5 a Parallel.  
   b Not parallel.  
   c Parallel  
   d Not parallel  
   e Parallel  
   f Not parallel  
   g Parallel

6 a i \(4p + 6 = -2p \Rightarrow p = -1\).  
   \(6 - 2q = 2 \Rightarrow q = 2\)
   ii \(3p + 2q = 7\).  
   \(-2q + p = 1 \Rightarrow 4p = 8 \Rightarrow p = 2\).  
   \(-2q = 1 - 2 = -1 \Rightarrow q = \frac{1}{2}\).

   b i \(\frac{p + 1}{2p} = k \Rightarrow p + 1 = 4k\).  
   \(2p = k \Rightarrow p + 1 = 8p \Rightarrow p = \frac{1}{7}\).
   ii \(\frac{2q - 3}{q + 6} = k \Rightarrow 2q - 3 = -3k\).  
   \(q + 6 = k \Rightarrow 2q - 3 = -3q - 18 \Rightarrow 5q = -15 \Rightarrow q = -3\).

Exercise 3I

1 a i \(a = \left(\begin{array}{c} 1 \\ 4 \end{array}\right)\).
ii \( AB = b - a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

iii \( AC = c - a = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \)

iv \( CA = -AC = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \)

b i \( BD = BA + AC + CD \)

ii \( BD = -AB + AC + CD \)

iii \( BD = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)

iv \( d = OB + BD = b + BD = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \).

2 a i \( AC = AB + BC = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \)

ii \( CA = \begin{pmatrix} 0 \\ -6 \end{pmatrix} \)

b \( DC = DA + AB + BC = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \)

3 a \( AB = b - a = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \). \( DC = c - d = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \).

b Parallelogram – two opposite sides have the same direction and length.

c \( AD \) and \( BC \)

4 a \( AC = \begin{pmatrix} 40 \\ -10 \end{pmatrix} \).

b \( \begin{pmatrix} -40 \\ 10 \end{pmatrix} \)

c \( d = \sqrt{40^2 + 10^2} = \sqrt{1700} = 41.2 \text{ km} \)

5 a Final displacement vector is \( (5 - 2 + 4 + 6)i + (1 + 4 + 2 + 4)j = 13i + 11j \).

b Displacement vector which will take him back to the start is \(-13i - 11j\).

Exercise 3J

1 a i Resultant is \( \begin{pmatrix} 4 \\ 4 \end{pmatrix} \). Magnitude is \( \sqrt{4^2 + 4^2} = 5.66 \)

ii Angle is \( \tan^{-1} \frac{4}{4} = 45^\circ \)
b  i  Resultant is \(-i - 2j\). Magnitude is \(\sqrt{1^2 + 2^2} = 2.24\)

ii  It is a good idea to always draw a diagram.

\[\text{Angle is } 180^\circ + \tan^{-1}\left(\frac{2}{1}\right) = 243^\circ\]

\[\text{Magnitude is } \sqrt{2^2 + 2^2} = 2.24\]

\[\text{Resultant is } \langle 2 \rangle.\]

\[\text{Angle is } \tan^{-1}\left(\frac{2}{1}\right) = 63.4^\circ\]

\[\text{Magnitude is } \sqrt{2^2 + 1^2} = 2.24\]

\[\text{Resultant is } \langle 8 \rangle.\]

\[\text{Angle is } \tan^{-1}\left(\frac{1}{8}\right) = 7.13^\circ\]

\[\text{Magnitude is } \sqrt{8^2 + 1^2} = 8.06\]

\[\text{Resultant is } \langle 8 \rangle.\]

\[\text{Angle is } \tan^{-1}\left(\frac{1}{8}\right) = 7.13^\circ\]

\[\text{Magnitude is } \sqrt{8^2 + 1^2} = 8.06\]

\[\text{Resultant is } \langle 2 \rangle.\]

\[\text{Angle is } \tan^{-1}\left(\frac{2}{1}\right) = 63.4^\circ\]

\[\text{Magnitude is } \sqrt{2^2 + 1^2} = 2.24\]

2  a  \(\frac{48}{20} = \sqrt{48^2 + 20^2} = \sqrt{2704} = \sqrt{16 \times 169} = 4 \times \sqrt{169} = 4 \times 13 = 52\)

b  i  \(\frac{18}{24} = \frac{6}{8} = \frac{3}{4} = 3 \times \frac{1}{4} = 3 \times \frac{1}{4} = 3 \times \frac{1}{2} = \frac{3}{2}\)

ii  \(-30 = 10 \times \frac{3}{4} = 10 \times 5 = 50.\)

iii  \(\frac{28}{-21} = \frac{7}{-3} = 7 \times \frac{1}{3} = 7 \times \frac{1}{3} = 7 \times 5 = 35.\)

3  a  \(\sqrt{3^2 + 4^2} = 5.\)

\(\frac{8}{5} \left(\frac{3}{4}\right) = \frac{1}{5} \left(\frac{24}{32}\right)\)

b  \(\sqrt{7^2 - 5^2} = \sqrt{49} = 7.\)

\(\left(\frac{5}{7}\right)\)

c  \(\sqrt{50} = \frac{k + 1}{k - 5} = \sqrt{(k + 1)^2 + (k - 5)^2} = \sqrt{2k^2 - 8k + 26} \Rightarrow 50 = 2k^2 - 8k + 26.\)
\[ k^2 - 4k - 12 = 0 = (k - 6)(k + 2). \] Since \( k > 0 \), the vector is \[ \begin{pmatrix} 7 \\ 1 \end{pmatrix}. \]

**4 a** North-east for 200 m: \( k \left\| \mathbf{b} \right\| = \sqrt{2}k = 200 \Rightarrow k = 100\sqrt{2} = 141.4. \)

Vector is: \[ \begin{pmatrix} 141 \\ 141 \end{pmatrix} \]

West for 175 m: \[ \begin{pmatrix} -175 \\ 0 \end{pmatrix}. \]

**b** \[ \begin{pmatrix} 141 - 175 \\ 141 \end{pmatrix} = \sqrt{21037} = 145 \text{ km} \]

**5** \[
\begin{pmatrix}
4 \sin(30) + 3 \sin(135) + 4 + 2 \sin(80) \\
4 \cos(30) + 3 \cos(135) + 0 + 2 \cos(80)
\end{pmatrix}
= \begin{pmatrix} 10.1 \\ 1.69 \end{pmatrix} = \sqrt{104.8661} = 10.2 \text{ km}.
\]

Bearing is \( \tan^{-1} \left( \frac{10.1}{1.69} \right) = 80.5^\circ \). So to travel back to the start has to travel at this bearing plus 180° so at a bearing of 261°.

**Exercise 3K**

**1 a** \[ \sqrt{8^2 + 4^2 + 1^2} = \sqrt{64 + 16 + 1} = \sqrt{81} = 9. \]

**b** \[ \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3. \]

**c** \[ 3\sqrt{4^2 + 3^2} = 3\sqrt{16 + 9} = 3 \times 5 = 15. \]

**d** \[ 5\sqrt{2^2 + 3^2 + 6^2} = 5\sqrt{4 + 9 + 36} = 5\sqrt{49} = 5 \times 7 = 35. \]

**2** \[ \begin{pmatrix} -12 \\ -9 \\ -6 \end{pmatrix}. \]

**3 a** To get from A to B, travel down the vector \( \mathbf{AB} = \mathbf{b} - \mathbf{a} \) so to travel from A to the midpoint of [AB] travel down \( \mathbf{AM} = \frac{1}{2} \mathbf{AB} = \frac{1}{2} (\mathbf{b} - \mathbf{a}) \). Now \( \mathbf{OM} = \mathbf{OA} + \mathbf{AM} = \mathbf{a} + \frac{1}{2} (\mathbf{b} - \mathbf{a}) \)

\[ \mathbf{OM} = \frac{1}{2} (\mathbf{a} + \mathbf{b}). \]
b \( PQ = q - p = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}, \quad QR = r - q = \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix} \)

c \( PR = PQ + QR = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix} \)

d \( PS = QR \). So S has the coordinates \((1 + 3, 3 + 4, 6 - 6) = (4, 7, 0)\).

e \( \frac{1}{2} PR = \begin{pmatrix} 0.5 \\ 0.5 \\ -3.5 \end{pmatrix} \)

f \( \frac{1}{2} QS = \frac{1}{2} (s - q) = \begin{pmatrix} 5 \\ 7 \\ -5 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 3.5 \\ -2.5 \end{pmatrix} \)

g The midpoint of the diagonal PR is \((1 + 0.5, 3 + 0.5, 6 - 3.5) = (1.5, 3.5, 2.5)\). Midpoint of the diagonal QS is \((-1 + 2.5, 0 + 3.5, 5 - 2.5) = (1.5, 3.5, 2.5)\). So the two diagonals bisect.

**Exercise 3L**

1 a \( \frac{a \cdot b}{|a||b|} = \frac{2 \times 3 - 1 \times 1 + 4 \times 2}{\sqrt{(2^2 + 1^2 + 4^2)(3^2 + 1^2 + 2^2)}} = \frac{13}{\sqrt{294}} \), \( \theta = \cos^{-1}\left(\frac{13}{\sqrt{294}}\right) = 40.7^\circ \).

b \( \frac{a \cdot b}{|a||b|} = \frac{2 \times -2 + 0 \times 1 + 1 \times -1}{\sqrt{(2^2 + 0^2 + 1^2)(2^2 + 1^2 + -1^2)}} = \frac{-5}{\sqrt{30}} \), \( \theta = \cos^{-1}\left(\frac{-5}{\sqrt{30}}\right) = 156^\circ \).

\( c \quad \frac{a \cdot b}{|a||b|} = \frac{2 \times 3 + 1 \times 2 - 1 \times 0}{\sqrt{(2^2 + 1^2 + 1^2)(3^2 + 2^2)}} = \frac{8}{\sqrt{78}} \), \( \theta = \cos^{-1}\left(\frac{8}{\sqrt{78}}\right) = 25.1^\circ . \)

\( d \quad \frac{a \cdot b}{|a||b|} = \frac{2 \times 3 - 1 \times 2 - 2 \times -5}{\sqrt{(2^2 + 1^2 + 2^2)(3^2 + 2^2 + 5^2)}} = \frac{14}{\sqrt{342}} \), \( \theta = \cos^{-1}\left(\frac{14}{\sqrt{342}}\right) = 40.8^\circ . \)

\( e \quad \frac{a \cdot b}{|a||b|} = \frac{2 \times 4 + 3 \times 6}{\sqrt{(2^2 + 3^2)(4^2 + 6^2)}} = \frac{26}{\sqrt{676}} \), \( \theta = \cos^{-1}\left(\frac{26}{\sqrt{676}}\right) = 0^\circ . \)

f \( \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \). So the vectors are parallel but in opposite directions. So the angle is 180°.

2 a i \( AC \) and \( AB \).

ii \( BC \) and \( BA \)
**Worked solutions**

\[ x = AB = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad y = BC = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad z = AC = \begin{pmatrix} 0 \\ -1 \\ -5 \end{pmatrix} \]

Angle at A:
\[
\frac{x \cdot z}{\|x\|\|z\|} = \frac{-1 \times 0 + 0 \times 1 + 2 \times -5}{\sqrt{(-1)^2 + 0^2 + 2^2}(0^2 + 1^2 + 5^2)} = \frac{-10}{\sqrt{130}}. \quad A = \cos^{-1}\left(\frac{-10}{\sqrt{130}}\right) = 151^\circ.
\]

Angle at B:
\[
\frac{y \cdot -x}{\|y\|\|-x\|} = \frac{1 \times 0 + 1 \times 0 + 0 \times -7 + 2 \times 0}{\sqrt{1^2 + 1^2 + 7^2}(1^2 + 0^2 + 2^2)} = \frac{15}{\sqrt{255}}. \quad B = \cos^{-1}\left(\frac{15}{\sqrt{255}}\right) = 20.1^\circ
\]

\[ C = 180 - 151 - 20.1 = 8.9^\circ. \]

**c** The longest side is the one opposite angle A, so is side BC. This length is the length of
\[ BC = \sqrt{1^2 + 1^2 + 7^2} = \sqrt{51} = 7.14 \]

3 **a** \[ a \cdot b = -6p + 2 - 2p = 0 \Rightarrow 8p = 2 \Rightarrow p = \frac{1}{4} \]

b
\[ a \cdot b = p(p - 1) - 2p - 4 = 0 \]
\[ p^2 - p - 2p - 4 = 0 \]
\[ p^2 - 3p - 4 = 0 \]
\[ (p - 4)(p + 1) = 0 \]
\[ p = -1, \quad 4. \]

4 **a**
\[ x = AC = \begin{pmatrix} 1 \\ 0 \\ k - 2 \end{pmatrix}, \quad y = BC = \begin{pmatrix} 0 \\ -1 \\ k + 1 \end{pmatrix} \]
\[ x \cdot y = (k - 2)(k + 1) = 0 \]

Hence, \( k = -1 \) or \( 2 \).

b When \( k = -1 \):
\[ AC = \sqrt{(1 - 2)^2 + (3 - 3)^2 + (2 - (-1))^2} = \sqrt{10}. \]
\[ BC = \sqrt{(2 - 2)^2 + (4 - 3)^2 + (-1 - (-1))^2} = 1. \]

Area of the triangle is \( \frac{\sqrt{10}}{2} \).

When \( k = 2 \):
\[ AC = \sqrt{(1 - 2)^2 + (3 - 3)^2 + (2 - 2)^2} = 1, \quad BC = \sqrt{(2 - 2)^2 + (4 - 3)^2 + (-1 - 2)^2} = \sqrt{10}. \]

Area of the triangle is \( \frac{\sqrt{10}}{2} \).

**5** Consider a unit cube that has one vertex at the origin and has one of its edges along the x-axis, one along the y-axis and the other along the z-axis. The vertex at the origin is \((0, 0, 0)\) and the diagonally opposite vertex is \((1, 1, 1)\) so the diagonal is the vector...
\[ a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

The diagonal between the vertices \((0,0,1)\) and \((1,1,0)\) is

\[
b = \begin{bmatrix} 0-1 \\ 0-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}
\]

\[
a \cdot b = \frac{1 \cdot -1 + 1 \cdot -1 + 1 \cdot 1}{\sqrt{(1^2 + 1^2 + 1^2)((-1)^2 + (-1)^2 + 1^2)}} = \frac{1}{3}, \quad \theta = \cos^{-1}\left(\frac{-1}{3}\right) = 109.47^\circ. \]

So the acute angle between two diagonals of a cube is \(180 - 109.47 = 70.5^\circ\).

**Exercise 3M**

1. \( a \times b = (1 \times 2) - (3 \times -1) = 5 \)

2. \( a \times b = \begin{bmatrix} -1 \times 2 + 1 \times 4 \\ 4 \times 3 - 2 \times 2 \\ 2 \times -1 - 3 \times -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix} \)

3. \( a \times b = (1 \times 0 - 1 \times 2) i + (-1 \times 3 - 2 \times 0) j + (2 \times 2 - 1 \times 3) k = 2i - 3j + k \)

4. \( a \times b = (-1 \times 5 - 2 \times 2) i + (-2 \times 3 - 2 \times -5) j + (2 \times 2 - 1 \times 3) k = 9i + 4j + 7k \)

5. \( BC = AC - AB = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \)

\[ \text{Area} = \frac{1}{2} |AB \times AC| = \frac{1}{2} \left| \begin{array}{ccc} 0 & -1 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 0 \end{array} \right| = \frac{1}{2} \left( \frac{-1}{2} \right) = \frac{1}{2} \sqrt{1^2 + 2^2} = \frac{\sqrt{5}}{2}. \]

\( AB = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad AC = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}. \)

Area of ABC is \(\frac{1}{2} |AB \times AC| = \frac{1}{2} \left| \begin{array}{ccc} 0 & -5 & -1 \\ 2 & 0 & 1 \\ 1 & -5 & 0 \end{array} \right| = \frac{1}{2} \left( \frac{-2}{2} \right) = \frac{\sqrt{30}}{2}. \)

5. \( AB = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}, \quad d = c - AB = \begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix} \)

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b  Area is $|AB \times AD| = \begin{vmatrix} 0 & 3 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{vmatrix} = 0 \times 0 - 8 \times 1 = -8 \times 3 - 0 \times 0 = 24 = 8 \times 3 = 8\sqrt{10}$.

c  $AB \cdot AD = 0 \times 3 + 0 \times 1 + 8 \times 0 = 0$.

6  $AB = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}, \quad AD = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}$.  $AB \cdot AD = -1 \times 6 + -3 \times -2 + 0 \times -1 = 6 - 6 = 0$.  So $AB$ and $AD$ are perpendicular sides of the roof.  So the total area is $|AB||AD| = \sqrt{1^2 + 3^2 + 1^2}\sqrt{6^2 + 2^2} = \sqrt{11} \times 4\sqrt{10} = 21.0 \text{ m}^2$.

7  The vector corresponding to the side of the base between the two given adjacent corners is $a = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$.  So the area of the base is $|a|^2 = a \cdot a = 3 \times 3 + 0 \times 0 + -2 \times -2 = 13$.  The vector corresponding to one of the edges from the vertex to the base is given by $b = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$.  So the area of one of the triangular sides is $\frac{1}{2} |a \times b| = \frac{1}{2} \begin{vmatrix} 0 \times -3 + 2 \times 0 \\ -2 \times 1 - 3 \times -3 \\ 3 \times 0 - 0 \times 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 0 \\ 7 \\ 0 \end{vmatrix} = \frac{7}{2}$.  So the total area is $13 + 4 \times \frac{7}{2} = 13 + 14 = 27$.

Exercise 3N

1  a  $r = \frac{2}{4} + t \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \frac{2}{4} + t \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \frac{2}{4} + t \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

   b  $r = \frac{2}{4} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \frac{2}{4} + t \begin{pmatrix} 4 - 2 \\ 2 - 1 \\ 0 - 1 \end{pmatrix} = \frac{2}{4} + t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

2  a  $2 + 8t = -2 \Rightarrow t = -\frac{1}{2}$.  $a = -\frac{1}{2} \times \frac{1}{1} = \frac{1}{2}, \quad b = 1 - \frac{1}{2} \times -2 = 2$.

   b  $0 = 1 + t \Rightarrow t = -1$.  $(1,2,1) - (1,8,-2) = (0,-6,3)$.

3  a  $2 + t \Rightarrow t = 1$.  $s = 0 + 1 \times 3 = 3, \quad p - 1 = 2 + 1 \times 1 \Rightarrow p = 4$.

   b  i  $3 + t = 1 \Rightarrow t = -2$.  $(3,1,2) - 2(1,2,-3) = (1,-3,8)$.

   ii  $AB = (5 - 2) \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \\ -21 \end{pmatrix}$.

4  a  $4 - t = 2 \Rightarrow t = 2$.  $(4,-5,1) + 2(-1,3,1) = (2,1,3)$.

   b  $4 + 2s = 2 \Rightarrow s = -1$.  $(4,2,0) - (2,1,-3) = (2,1,3)$.
Worked solutions

\[\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \frac{-1 \times 2 + 3 \times 1 + 1 \times -3}{\sqrt{(1^2 + 3^2 + 1^2)(2^2 + 1^2 + 3^2)}} = -\frac{2}{\sqrt{11 \times 14}}. \quad \theta = \cos^{-1}\left( -\frac{2}{\sqrt{154}} \right) = 99.27. \] So acute angle is \(180 - 99.27 = 80.7^\circ\).

5 a \(l_1\) and \(l_2\) are parallel as the direction vectors for each line are scalar multiples of each other.

b \(3 - t = 1 \Rightarrow t = 2, \quad (3, 5, 2) + 2(-1, 2, 1) = (1, 9, 4)\)

\(3 + 2s = 1 \Rightarrow t = -1, \quad (3, 5, 2) - (2, -4, -2) = (1, 9, 4)\)

c This tells us that the two lines are the same line.

6 a \(3 + s = 1 \Rightarrow s = -2, \quad (3, 1, 2) - 2(1, 3, -2) = (1, -5, 6)\).

b \(AB = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \quad AB \cdot \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = 1 \times 4 + 2 \times 3 + 5 \times -2 = 0.\)

7 a \((2 + t, 3t, 1 + 2t)\).

b \(AP = \begin{pmatrix} 1 + t \\ 3t - 2 \\ 3 + 2t \end{pmatrix}\)

c \[\begin{align*}
AP \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} &= 0 \\
AP \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} &= 1(1 + t) + 3(3t - 2) + 2(3 + 2t) \\
&= 1 + t + 9t - 6 + 6 + 4t \\
&= 14t + 1 \Rightarrow t = -\frac{1}{14}.\]

d Point on \(l_1\) that is closest to \(A\) is the point with \(t = -\frac{1}{14} : \quad (2, 0, 1) - \frac{1}{14}(1, 3, 2) = \left(\frac{27}{14}, -\frac{3}{14}, \frac{6}{7}\right).\)

e Shortest distance from \(A\) to \(l_1\) is \(\sqrt{\left(\frac{13}{14}\right)^2 + \left(\frac{31}{14}\right)^2 + \left(\frac{20}{7}\right)^2} = \frac{1}{14} \sqrt{2730} = 3.73\)

Exercise 30

1 a i \(n = \frac{1}{2}((4 - 2)i + (5 - 1)j) = i + 2j\)
ii \( |v| = \sqrt{1^2 + 2^2} = \sqrt{5}. \)

\[ b \]

i \( v = \frac{1}{4} \begin{pmatrix} 1 - 2 \\ 1 - 1 \end{pmatrix} = \begin{pmatrix} -0.25 \\ 0.5 \end{pmatrix}. \)

ii \( |v| = \sqrt{0.25^2 + 0.5^2} = 0.559 \)

\[ c \]

i \( v = \frac{1}{2} ((1 - 3)i + (5 - 1)j + (1 - 1)k) = -i + 2j + k. \)

ii \( |v| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}. \)

\[ d \]

i \( v = \frac{1}{4} \begin{pmatrix} 1 - 1 \\ 4 - 0 \\ -3 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \)

ii \( |v| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}. \)

\[ 2 \]

\[ a \]

\( \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \end{pmatrix}. \)

\[ b \]

\( 7.5 \begin{pmatrix} -3 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -4.5 \\ 6 \end{pmatrix}. \)

\[ c \]

\( \begin{pmatrix} -1 \\ 1 \\ 8 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \\ 16 \end{pmatrix}. \)

\[ d \]

\( \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3.54 \\ -3.54 \end{pmatrix}. \)

\[ e \]

Anticlockwise angle with x-axis is \( 90 - 40 = 50^\circ \).

\[ \begin{pmatrix} 15\cos(50) \\ 15\sin(50) \end{pmatrix} = \begin{pmatrix} 9.64 \\ 11.5 \end{pmatrix} \]

\[ f \]

Clockwise angle with x-axis is \( 120 - 90 = 30^\circ \)
12\cos(30)

\begin{bmatrix}
12\cos(30) \\
-12\sin(30)
\end{bmatrix}

= \begin{bmatrix}
10.4 \\
-6
\end{bmatrix}.

\[ 3 \; \text{a} \quad \sqrt{20^2 + 30^2} = 36.1 \text{m} \]

\[ 3 \; \text{b} \quad p = \begin{bmatrix} 20 \\ 30 \end{bmatrix} + t \begin{bmatrix} -3 \\ -5 \end{bmatrix}. \]

\[ 3 \; \text{c} \quad d = \sqrt{(20 - 3t)^2 + (30 - 5t)^2}. \text{ From the GDC this is minimized when } t = 6.18 \text{ seconds and the shortest distance is 1.71 m.} \]

\[ 4 \; \text{a} \quad p = 3i + j + t \frac{10(3i - 4j)}{|3i - 4j|} = 3i + j + t(6i - 8j). \]

\[ 4 \; \text{b} \quad (3 + 6 \times 4)i + (1 - 8 \times 4)j = 27i - 31j. \]

\[ 4 \; \text{c} \quad \sqrt{27^2 + 31^2} = 41.1 \text{m} \]

\[ 4 \; \text{d} \quad 4 \times 10 = 40 \text{ m} \]

\[ 5 \; \text{a} \quad (1, 0, 2) + 3(-1, 3, 1) = (-2, 9, 5) \]

\[ 5 \; \text{b} \quad \text{i} \quad r = \begin{bmatrix} -2 \\ 9 \\ 5 \end{bmatrix} + (t - 3) \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \]

\[ 5 \; \text{ii} \quad r = \begin{bmatrix} -2 \\ 9 \\ 5 \end{bmatrix} + t' \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}. \]

\[ 5 \; \text{c} \quad \text{i} \quad (-2, 9, 5) + 2(1, 4, 0) = (0, 17, 5). \]

\[ 5 \; \text{ii} \quad \sqrt{17^2 + 5^2} = 17.7 \]

\[ 6 \; \text{a} \quad \text{Compare i coefficients: } (-5 + 2t) = (3 - 2t) \Rightarrow t = 2. \text{ At } t = 2 \text{ the j coefficients are } 10 + 2(2) = 14 \text{ and } 4 + 2(5) = 14. \text{ So the two ships collide when } t = 2 \text{ (at 12:00).} \]

\[ 6 \; \text{b} \quad a = (-5 + t)i + (10 + 2t)j. \]

\[ AB = b - a = (3 - 2t - t + 5)i + (4 + 5t - 10 - 2t)j \]

\[ = (8 - 3t)i + (3t - 6)j \]

\[ 6 \; \text{c} \quad \text{A is north of B when } (8 - 3t) = 0 \Rightarrow t = \frac{8}{3}. \text{ So at 12:40. The distance is } 3 \left( \frac{8}{3} \right) - 6 = 2 \text{ km}. \]

\[ 6 \; \text{d} \quad d = \sqrt{(8 - 3t)^2 + (3t - 6)^2}. \text{ From the GDC this is minimized when } t = 2.33. \text{ So at 12:20. At this time the distance is 1.41 km.} \]
7 a i \[
\frac{\begin{pmatrix} 6 & 2 & 3 \\ 2 \\ 3 \end{pmatrix}}{6} = \begin{pmatrix} 3 \\ 1 \\ 1.5 \end{pmatrix}.
\]

ii \[1.5\text{ms}^{-1}\]

b \[
\begin{pmatrix} 0 \\ 0 \\ 30 \end{pmatrix} + 60 \begin{pmatrix} 1 \\ 1.5 \\ 120 \end{pmatrix} = \begin{pmatrix} 180 \\ 60 \\ 120 \end{pmatrix}.
\]

c \[r = \begin{pmatrix} 180 \\ 60 \\ 120 \end{pmatrix} + t' \begin{pmatrix} 3 \\ 1 \\ -0.6 \end{pmatrix}.
\]

d \[
\frac{120}{0.6} = 200\text{s}
\]

e \[
\begin{pmatrix} 180 + 200 \times 3 \\ 60 + 200 \times 1 \\ 0 \end{pmatrix} = \sqrt{780^2 + 260^2} = 822\text{m}.
\]

8 a \[
\begin{pmatrix} 0 \\ -9 \end{pmatrix}.
\]

\[
p_S = \begin{pmatrix} 20 \\ 15 \end{pmatrix} + \begin{pmatrix} 0 \\ -9 \end{pmatrix}t
\]

\[
p_B = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 9 \\ 12 \end{pmatrix}t
\]

b \[
\sqrt{(20 - 0)^2 + (15 - 5)^2} = 10\sqrt{5} = 22.4\text{km}
\]

c \[BS = s - b = \begin{pmatrix} 20 - 9t \\ 15 - 9t - 5 - 12t \end{pmatrix} = \begin{pmatrix} 20 - 9t \\ 10 - 21t \end{pmatrix}
\]

d Unit vector in south-east direction is \[
\begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\] BS is in the same direction as this vector, i.e.

\[
\tan \theta = \frac{10 - 21t}{20 - 9t} = -\frac{1}{1}
\]

\[-(20 - 9t) = 10 - 21t \Rightarrow t = 1.
\]

Boat changes direction at 11:00. Displacement is \[
\begin{pmatrix} 9 \\ 17 \end{pmatrix}.
\]

\[
e \begin{pmatrix} 20 \\ 6 \end{pmatrix}.
\]
The speed of the boat before changing direction was \( \sqrt{9^2 + 12^2} = 15 \)

If the direction of the boat after the change is at an angle of \( \theta \), clockwise from east then its position is given by \( r = \begin{pmatrix} 9 \\ 17 \end{pmatrix} + 15t \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \) where \( t' \) is the time after it changes direction. The ship’s position is given by \( s = \begin{pmatrix} 20 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ -9 \end{pmatrix} t' \).

When the boat intercepts the ship

\[
20 = 9 + 15t' \cos(\theta) \Rightarrow t' = \frac{11}{15 \cos(\theta)}.
\]

\[
6 - 9t' = 17 - 15t' \sin(\theta) \Rightarrow 15t' \sin(\theta) = 11 + 9t'.
\]

Since \( t' = \frac{11}{15 \cos(\theta)} \), we can substitute to get

\[
\frac{11 \sin(\theta)}{\cos(\theta)} = 11 + \frac{99}{15 \cos(\theta)} \Rightarrow 11 \sin(\theta) - 11 \cos(\theta) = \frac{99}{15} \Rightarrow \sin(\theta) - \cos(\theta) = \frac{3}{5}
\]

Solve on a GDC to obtain, \( \theta = 70.1^\circ \), hence bearing is 160.1°.

**Chapter review**

1. **a** \( \sqrt{(1.2 + 0.2)^2 + (8.5 - 9.4)^2 + (3.1 - 2.6)^2} = 1.74 \text{km} \)

   **b** First aircraft: \( \sqrt{1.2^2 + 8.5^2 + 3.1^2} = 9.13 \text{km} \)

   Second aircraft: \( \sqrt{0.2^2 + 9.4^2 + 2.6^2} = 9.75 \text{km} \). Second aircraft is further.

2. **a** \( m = \frac{3-5}{6-1} = -\frac{2}{5} = -0.4 \quad y = -0.4x + c \quad 5 = -0.4 + c \Rightarrow c = 5.4 \quad y = -0.4x + 5.4 \)

   **b** \( 5x - 2(-0.4x + 5.4) + 5 = 0 \Rightarrow 5.8x = 5.8 \Rightarrow x = 1, \quad y = 5. \quad \text{(1,5).} \)

3. **a** Perpendicular bisector of \( [AB] \) is \( y = 4 \) and of \( [BC] \) is \( x = 5 \).
b Midpoint of [AC] is \( \left( \frac{2 + 8}{2}, \frac{4 + 6}{2} \right) = (5,5) \). \( m = \frac{8 - 2}{6 - 4} = -3 \). \( y = -3x + 20 \)

c

d Midpoint of [AD] is \( \left( \frac{2 + 4}{2}, \frac{4 + 3}{2} \right) = (3,3.5) \). \( m = \frac{4 - 2}{3 - 4} = 2 \). \( y = 2x - 2.5 \).

e

4 a Perpendicular bisector of [BC]: Midpoint \( \left( \frac{6 + 3}{2}, \frac{5 + 2}{2} \right) = (4.5,3.5) \). \( m = \frac{6 - 3}{5 - 2} = -1 \).

\[ y = -x + 8. \]

Perpendicular bisector of [AC]: Midpoint \( \left( \frac{2 + 3}{2}, \frac{4 + 2}{2} \right) = (2.5,3) \). \( m = \frac{-2 - 3}{4 - 2} = \frac{1}{2} \).

\[ y = \frac{1}{2}x + \frac{7}{4}. \]

b Should be placed at the intersection of the perpendicular bisectors: \( \frac{1}{2}x + \frac{7}{4} = -x + 8 \)

\[ \Rightarrow \frac{3}{2}x = \frac{25}{4} \Rightarrow x = 4.17, y = 3.83. \] Should be built at (4.17,3.83).

c Distance is the same to each of the previous three outlets and is

\[ \sqrt{(2 - 4.17)^2 + (4 - 3.83)^2} = 2.18 \text{ km} \]

5 a \( t = 0: \left\{ \frac{5}{6} \right\} \). \hspace{1cm} b \( t = 2: \left\{ \frac{-3}{0} \right\} \).

c \( \sqrt{4^2 + 3^2} = 5 \text{ m min}^{-1} \).

d \( \sqrt{(5 - 4t)^2 + (6 - 3t)^2} \). From the GDC the minimum of this is when \( t = 1.52 \text{ min} \). The distance is 1.8 m.
6 a
\[
\mathbf{a} \cdot \mathbf{b} = 1 \times 2 + 2 \times (q - 1) + p \times 1 \\
= 2 + 2q - 2 + p \\
= 2q + p
\]
2q + p = 0.

b
\[
b = ka \Rightarrow 2 = k \times 1 \Rightarrow k = 2
\]
q - 1 = 2 \times 2 ⇒ q = 5
1 = 2 \times p ⇒ p = 0.5

c
\[
a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}
\]
\[
\mathbf{a} \cdot \mathbf{b} = \frac{(1 \times 2) + (2 \times 1) + (3 \times 1)}{(1^2 + 2^2 + 3^2)(2^2 + 1^2 + 1^2)} = \frac{7}{\sqrt{84}} \quad \theta = \cos^{-1}\left(\frac{7}{\sqrt{84}}\right) = 40.2^\circ.
\]

7 Two sides of the base triangle are given by the vectors \(BA = a - b = \begin{pmatrix} -2.5 \\ -2.0 \\ 0 \end{pmatrix}\) and \(CA = a - c = \begin{pmatrix} 1.0 \\ -1.5 \\ 0 \end{pmatrix}\) so the area of the triangular base is
\[
\frac{1}{2} \begin{vmatrix} 2.5 & 1 & 0 \\ -2 & -1.5 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} -2 \times 0 - 0 \times -1.5 \\ 1 \times 0 - 2.5 \times 0 \\ 2.5 \times -1.5 - 1 \times -2 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} -0 \\ 0 \\ -1.75 \end{vmatrix} = 0.875. \text{ The height of the tetrahedron is 2 so} \\
\text{the volume is } \frac{1}{3} \times 2 \times 0.875 = 0.583.
\]

8 a
\[-8 = 3 - t \Rightarrow t = 11. \quad n = 2 + 2t = 2 + 22 = 24.\]

b
\[1 - s = -8 \Rightarrow s = 9.\]
\[
\text{Hence, } \begin{pmatrix} 1 \\ -2 \\ u \end{pmatrix} + 9 \begin{pmatrix} -1 \\ p \\ q \end{pmatrix} = \begin{pmatrix} -8 \\ 34 \\ 24 \end{pmatrix}
\]
\[-2 + 9p = 34 \Rightarrow p = 4.
\]
\[
\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 1 + 12 + 2q = 0 \Rightarrow q = -6.5
\]
\[ u + 9q = 24 \]
\[ u = 24 - 9 \times (-6.5) = 24 + 58.5 = 82.5. \]

\[ p = 4, \quad q = -6.5, \quad u = 82.5 \]

\[ 9 \quad a \quad r_A = \begin{bmatrix} 0 \\ 8.2 \end{bmatrix} + t \begin{bmatrix} 750 \cos 45 \\ 750 \sin 45 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8.2 \end{bmatrix} + t \begin{bmatrix} 530.3 \\ 530.3 \end{bmatrix}. \]

\[ b \quad r_B = \begin{bmatrix} 0 \\ 13.1 \end{bmatrix} + (t - 0.5) \begin{bmatrix} 800 \sin (30) \\ 800 \cos (30) \end{bmatrix} = \begin{bmatrix} 0 \\ 13.1 \end{bmatrix} + (t - 0.5) \begin{bmatrix} 400 \\ -1 \end{bmatrix}. \]

\[ c \quad 8.2 + 2t = 13.1 - (t - 0.5) \Rightarrow 3t = 5.4 \Rightarrow t = 1.8. \]

\[ r_A = \begin{bmatrix} 0 \\ 8.2 \end{bmatrix} + 1.8 \times \begin{bmatrix} 530.3 \\ 530.3 \end{bmatrix}. \]

\[ r_A = \begin{bmatrix} 0 \\ 13.1 \end{bmatrix} + 1.3 \times \begin{bmatrix} 400 \\ 693 \end{bmatrix}. \]

Ignoring the height,

\[ d = \sqrt{(1.8 \times 530.3 - 1.3 \times 400)^2 + (1.8 \times 530.3 - 1.3 \times 693)^2} = 438\text{km}. \]

**Exam style Questions**

**10 a**

\[ M \text{ is } \begin{bmatrix} -3 + 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 + 3 \\ 2 \end{bmatrix} = (1, 5.5). \]

\[ b \quad m = \frac{3 - 8}{5 - 3} = \frac{5}{8} = -0.625. \]

\[ c \quad i \quad \frac{8}{5} = 1.6. \]

\[ ii \quad y = 1.6x + c. \quad 5.5 = 1.6 + c \Rightarrow c = 3.9. \quad y = 1.6x + 3.9. \]

**11 i**

Neither, as the gradients are not the same and their product is not \(-1\).

**ii**

Parallel, as both lines have a gradient of 3.

**iii**

Neither, as the gradients are not the same and their product is not \(-1\).

**iv**

Perpendicular, as the product of the gradients is \(-1\).

**v**

Perpendicular, as the product of the gradients is \(-1\).

**12 a**

\[ \sqrt{500^2 + 400^2 + 300^2} = 707\text{m} \]

\[ b \quad 707.10 \ldots + \sqrt{400^2 + 200^2 + 400^2} = 1310\text{m} \]
**13 a** \[ AB = b - a = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad AC = c - a = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad AB \times AC = \begin{pmatrix} 1 \times 1 - 0 \times 3 \\ 0 \times 3 - 1 \times 1 \\ 1 \times 1 - 3 \times 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \]

**b** The area is \[ \frac{1}{2} |AB \times AC| = \frac{1}{2} \begin{vmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{vmatrix} = \frac{1}{2} \times \sqrt{2} = \frac{\sqrt{2}}{2}. \]

**14 a** \[ \frac{a}{3} \times \frac{2}{3} = -1 \Rightarrow a = -4.5. \]

**b**

\[-4.5x - 3 \left( \frac{2}{3}x + 4 \right) = 9 \]
\[-6.5x - 12 = 9 \]
\[-6.5x = 21 \]
\[x = -3.23 \]
\[y = \frac{2}{3}x + 4 = 1.85. \]

(-3.23, 1.85).

**15 a** \[ AB = 5, \quad AF = 6, \quad AO = 3. \quad \text{Surface area is } 2 \times (5 \times 6 + 5 \times 3 + 3 \times 6) = 126. \]

**b** \[ \sqrt{3^2 + 5^2 + 6^2} = \sqrt{70} = 8.37. \]

**c**

i. \( \left( \frac{3 + 0}{2}, \frac{5 + 0}{2}, \frac{6 + 0}{2} \right) = (1.5, 2.5, 3). \)

ii. \[ x = AM = m - a = \begin{pmatrix} -1.5 \\ 2.5 \end{pmatrix}, \quad y = BM = m - b = \begin{pmatrix} -2.5 \\ 3 \end{pmatrix}. \]

\[ x \cdot y = \frac{-1.5 \times -1.5 + 2.5 \times -2.5 + 3 \times 3}{\sqrt{(1.5^2 + 2.5^2 + 3^2)(1.5^2 + 2.5^2 + 3^2)}} = \frac{5}{17.5} = 0.285. \]

\[ \theta = \cos^{-1} \left( \frac{5}{17.5} \right) = 73.4^\circ. \]

**16 a** The distance to the town from a point \((x, y)\) on the road is given by

\[ d = \sqrt{(80 - x)^2 + (140 - y)^2} = \sqrt{(80 - x)^2 + (140 - x + 80)^2} = \sqrt{2x^2 - 600x + 54800}. \]

This is minimized when \(x = \frac{600}{2} = 150. \quad y = 150 - 80 = 70. \quad \text{The new airport is at (150, 70).} \]

**b** The distance between the town and airport is \[ \sqrt{(80 - 150)^2 + (140 - 70)^2} = 99.0 \text{ km}. \]
17 a

\[b \quad \frac{1}{4} \pi \times 10^2 = 78.5 \text{ m}^2 \]

c

\[
d \quad 5 \times 5 = 25 \text{ m}^2. 
\]

e \quad \frac{1}{4} \pi (\pi \times 10^2 - 25) = \frac{1}{4} \times 289.16 = 72.3 \text{ m}^2.

f \quad 4.

18 a

\[
\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 50 \\ 60 \\ 1 \end{pmatrix} = \begin{pmatrix} 200 \\ 240 \\ 5 \end{pmatrix}. 
\]

\[
b \quad 1 + t = \lambda \Rightarrow s = \begin{pmatrix} -90 \\ -100 \\ 0 \end{pmatrix} + (1 + t) \begin{pmatrix} 60 \\ 70 \\ 1 \end{pmatrix} = \begin{pmatrix} -30 \\ -30 \\ 1 \end{pmatrix} + t \begin{pmatrix} 60 \\ 70 \\ 1 \end{pmatrix}. 
\]

\[
gt = -30 + 60t \Rightarrow 10t = 30, \Rightarrow t = 3. 
\]

When \( t \) is 3 the first aircraft is at \( \begin{pmatrix} 150 \\ 180 \\ 4 \end{pmatrix} \), and when \( \lambda \) is 4 the second aircraft is at \( \begin{pmatrix} 150 \\ 180 \\ 4 \end{pmatrix} \),

so the two flightpaths do cross and cross at \( \begin{pmatrix} 150 \\ 180 \\ 4 \end{pmatrix} \).

c The two aircrafts do not collide as the first aircraft gets to the intersection point an hour before the second does.

d \quad \sqrt{90^2 + 100^2 + 1^2} = \sqrt{18101} = 135 \text{ km}.

e The distance between the aircraft is given by

\[
d = \sqrt{(50t + 90 - 60t)^2 + (60t + 100 - 70t)^2 + (1 + t - t)^2} = \sqrt{(90 - 10t)^2 + (100 - 10t)^2 + 1^2}.
\]

This is minimized when \( t = 9.5 \) and at this time the distance is 7.14 km. So the shortest distance between the planes is 7.14 km after 9 and a half hours.
Paper 3

a  
  i  Coordinates of B are \( \left( 6370\cos 50^\circ, 6370\sin 50^\circ \right) \)  
    \[ 4090, 4880 \]  
    M1A1

  ii  \[ \frac{50}{360} \times 2\pi \times 6370 = 5560 \]  
    M1A1

  iii  \[ \sqrt{(6370 - 4090)^2 + 4880^2} \]  
    \[ = 5380 \text{ km} \]  
    A1

  iv  Percentage error = \[ \frac{5560 - 5380}{5560} \times 100 = 3.2\% \]  
    M1A1

  [9 marks]

b  
  i  \( R \sin \theta \)  
    A1

  ii  Projection of \( P \) onto the plane of the equator is a distance \( R \cos \theta \) from O (M1)  
    Coordinates are \( \left( R \cos \theta \cos \phi, R \cos \theta \sin \phi, R \sin \theta \right) \)  
    A1A1

  [4 marks]

c  
  i  \( \left( 6370\cos(-12.05)\cos(-77.04), 6370\cos(-12.05)\sin(-77.04), 6370\sin(-12.05) \right) \)  
    M1A1

    \[ = \left( 1397, -6071, -1329 \right) \]  
    AG

  [2 marks]
d i 6370 km A1

1397 3945

ii 6071 3346 3076424 M1(A1)

1329 3717

Using the fact that both cities lie on the surface of the earth, and hence have magnitude 6370, gives

\[ \cos \left( \hat{LOP} \right) = \frac{-3076424}{6370^2} = -0.758 \] M1A1

= 139.3° A1

Note: It is also possible to solve ii by finding the straight-line distance between Lima and Tokyo, and use the cosine rule or right-angled trigonometry.

iii \[ \frac{139.3}{360} \times 2\pi \times 6370 \] (M1)

= 15 490 km A1

[8 marks]

Total: 27 marks
Skills check

1. a \( y = 0.25x + 1.75 \) 
   b \( y = -3x - 4 \)

2. a \( y = 23 \) 
   b \( x = 4 \)

Exercise 4A

1. a Not a function, because one x-value may have multiple y-values. As there are 500 people and 12 months, at least 1 month will have two or more people with a birthday in it.
   b \( y \) is a function of \( x \), because for each month, there is exactly one value for the number of people with a birthday in it. For example, if 42 people have birthdays in January, then \( x = 1 \) corresponds to \( y = 42 \).

2. a \( R_3 \) is a function. Every element from the first set, \( A \), is mapped onto only one element from the second set, \( B \).
   b Not a function. In \( R_4 \), the input 3 has two outputs, 1 and 2.
   c Not a function as it fails the vertical line test
   d Is a function, passes the vertical line test
   e Is a function, for each \( x \) value, there is a unique \( y \) value
   f Not a function as it fails the vertical line test.

3. a \( R : x \rightarrow \frac{1}{x} \)
   b Yes.
   c \( \frac{1}{2} \)
   d \( a = \frac{1}{3} \)

4. a \((-1)^3 = -1\)
   b \( x^3 = -64 \)
   \( x = \sqrt[3]{-64} \Rightarrow x = -4 \)
   c \( B = \{-1, 0, 1, 8, 27, 64\} \)
   d It is a function, because for each \( x \) value, there is a unique \( y \) value.

Exercise 4B

1. a i 5  
   ii −3
   b i 0  
   ii 3
   c \( x > 3 \)
2 a i \( f(2) = 10 - 4(2) = 2 \)

ii \( f\left(-\frac{1}{2}\right) = 10 - 4\left(-\frac{1}{2}\right) = 12 \)

b \( f(2.5) = 10 - 4(2.5) = 0 \)

c \( 10 - 4x = -6 \)
\(-4x = -16 \)
\(x = 4 \)

3 a i \( F(0) \approx 687 \text{ N} \). This is Jaime’s weight at sea level.

ii \( F(410) \approx 607 \text{ N} \)

iii \( \frac{607}{687} \times 100 = 88.4\% \)
\(100\% - 88.4\% = 11.6\% \)
Jaime is 11.6\% lighter on the space station.

b When \( F = 625, h = 310 \text{ km} \). The force of gravity on Jaime is 625 N at a height of 310 km above sea level.

c \( 0.95 \times 687 = 653 \)
\( F = 653 \) when \( h \approx 170 \text{km} \) above sea level.
\( F(653) = 170 \)

Exercise 4C

1 a On January 2nd last year the average temperature was 25°C.

b Domain = \{1, 2, 3, ..., 31\}

c Answers need to include all the values in the table so should be similar to \( 20 \leq T \leq 30 \), or \( T \in [20,30] \) if it can take any value

2 a i \( f(-1) = -2(-1) + 3 = 5 \) (ii) \( f(3) = -2(3) + 3 = -3 \)

b \(-2x + 3 = 2 \)
\(-2x = -1 \)
\(x = \frac{1}{2} \)

c
d $-3 \leq y \leq 5$ or $y \in [-3, 5]$

3 a Domain = \{-5, -4, -3, -2, -1, 0, 1, 2, 4\}
Range = \{-2, 0, 2, 4, 6, 8\}

b Domain : $-8 \leq x \leq 6$
Range: $-4 \leq y \leq 3$

c Domain: $-7 < x \leq 9$
Range: $0 \leq y < 4$

d Domain: $-7 < x < 7$
Range: $-4 \leq y < 3$

4 a $p \in \{3, 4, 5, 6, 7, 8\}$

b $w(3) = 9 + \frac{15}{3+1} = 12.8\text{kg}$, $w(8) = 9 + \frac{15}{8+1} = 10.7\text{kg}$. If there are 3 people in the group, each must carry 12.8 kg of food. If there are 8 people in the group, each must carry 10.7 kg of food.

c The function goes from $(3, 12.8)$ to $(8, 10.7)$, so the range is $\{w | 10.7 \leq w \leq 12.8\}$

d
\[
11 = 9 + \frac{15}{p+1}
\]
\[
p = \frac{13}{2} = 6.5
\]
So, $p \leq 6$.

\[
12 = 9 + \frac{15}{p+1}
\]
\[
p = 4
\]
Hence, $p \in \{4, 5, 6\}$. So, Robbin can take from 4 to 6 people.

5 a $0 \leq t \leq 5$ as valid up to January 2019

b $S(0) = $3.81 (given)

\[
S(5) = -0.09(5)^2 - 0.0651(5) + 3.81 = $1.23
\]
\[
1.23 \leq S \leq 3.81
\]

c $S(2.5) = -0.09(2.5)^2 - 0.0651(2.5) + 3.81 = $3.08$

d $-0.09t^2 - 0.0651t + 3.81 = 1.5$
\[
-0.09t^2 - 0.0651t + 2.31 = 0
\]
\[
t = 4.72
\]
During 2018
The researchers would be extrapolating far beyond the data set; also the values of the function become negative after 6 years.

6 a  \[ R(2500) = \frac{1.75}{10000} (2500)(30000 - 2500) \approx \€12031 \]

b  \[ R(5000) = \frac{1.75}{10000} (5000)(30000 - 5000) \approx 21875 \]

\[ 21875 - 12031 = \€9844 \]

c  \[ 27000 - \frac{60000000}{q + 5000} = 20000 \]

\[ \frac{60000000}{q + 5000} = 7000 \]

\[ q + 5000 = \frac{60000000}{7000} \]

\[ q = 3571 \]

The cost to produce 3571 bottles is €20000.

d

![Graph]

One break-even point is at \( q = 4702 \) and the other is at \( q = 24083 \)

e  Company makes a profit when \( q \vert 4702 \leq q \leq 24083 \)

7 a  \[ V(0) = 8500 + 5(-15)^2 = 9625, \ a = \£9625 \]

\[ V(1) = \frac{8500}{2} + 5(-13)^2 = 5230, \ b = \£5230 \]

b  Technology shows that the value drops below after 9.2 years so during 2017

c  She bought it for \£9625. According to technology, it will be worth that again after 58.5 years, so during 2066.

**Exercise 4D**

1 a  linear with gradient zero

b  linear with gradient \(-2\)

c  not linear

d  linear with gradient \(5\)

2 a  i  independent = time parked (in hours)

dependent = cost of parking (in $)
ii Linear, with a constant rate of change of $12.50 per hour

b i independent = time (in years)
   dependent = population (in number of fish)

ii Not linear, as a decline of 7% of the current population will not be a constant number of fish per year

c i independent = amount of purchase (euros)
   dependent = amount of tax (euros)

ii Linear, with a constant rate of change of €0.22 tax per euro spent

d i independent = daily high temperature (in °C)
   dependent = number of daily passes sold (number of passes)

ii The relationship is linear, because the rate of change between any two points of the function is constant and equivalent to a decrease of 8 passes per degree Celsius.

3 It is not a linear function. The rate of change from 1 to 3 is \( \frac{4}{2} = 2 \). The rate of change from 5 to 8 is \( \frac{4}{3} \neq 2 \)

4 a US$30
   b \( P(1.5) = 100(1.5) + 30 = \) US$180
   c \( 100x + 30 = 310 \)
      \( 100x = 280 \)
      \( x = 2.8 \text{ kg} \)

5 a \( d(t) = 13 - 0.065t \)
   b \( d(60) = 13 - 0.065(60) = 9.1 \text{ km} \)
   c \( 13 - 0.065t = 0 \)
      \( 0.065t = 13 \)
      \( t = 200 \)

   Ewout will take 200 minutes or approximately 3 and a half hours to reach home.

6 a 64% – 62.65% = 1.35%
   \( S(t) = 64 - 1.35t \)

b 64 – 1.35t = 50
   \( 1.35t = 14 \)
   \( t = 10.37 \)

   10.37 years after 2018, so during 2028

c \( \frac{18-13}{2-0} = 2.5 \)

   Foot Talker grows at 2.5% of total sales per year, whereas Sneakies declines by 1.35% of total sales.

d \( F(t) = 13 + 2.5t \)

e \( 13 + 2.5t = 64 - 1.35t \)
   \( 3.85t = 51 \)
   \( t = 13.2 \)
13.2 years after 2018, so during 2031

7  a 120 m s\(^{-1}\)  b 8 sec  c \(\frac{-120}{8} = -15\) m s\(^{-2}\)

d \(v(t) = -15t + 120\)

8  a 50m + c = 20  b 80m + c = 35

c \(30m = 15\)
\(m = 0.5\)
\(50(0.5) + c = 20\)
\(c = 20 - 25 = -5\)
\(L = 0.5W - 5\)

d \(L = 0.5(90) - 5 = 40\) cm

Exercise 4E

1  a \(c = \frac{1}{0.21}\) or \(c = 4.76a\)

b one AUD is worth 4.76 CNY

c \(599 = 4.76a\)
\(a = 125.84\)
\(126 - 75 = 51\)

51 AUD needed

2  a 15 = 0.64k
\(k = 23.4\)
\(F(x) = 23.4x\)

b \(80 = 23.4x\) \(x = 3.42\) m

c \(23.4(-1.5) = -35.1\) N

Exercise 4F

1  a \(u(x) = 1.33x\)

b \(1.33x = 100\)
\(x = \£75.19\)

c i \(u(500) = 665,\ 665 - 661.72 = 3.28\)

ii \(B(x) = 1.33x - 3.28\)

iii \(1.33x - 3.28 = 1000\)
\(1.33x = 1003.28\)
\(x = \£754.35\)

2  a gradient: for each 1 euro that the price increases, the number of people willing to buy tickets decreases by 36.
y-intercept: 5000 people will buy the tickets if they cost 0 euros.

b $N(75) = 5000 - 36(75) = 2300$

c $5000 - 36p = 0$
   
   $36p = 5000$
   
   $p = 138.89$ euros

This is the price at which no one is willing to buy a ticket.

d Domain $\{p \mid 0 \leq p \leq 139\}$, range $\{N \mid 0 \leq N \leq 5000\}$

e $5000 - 36p = 28p - 504$
   
   $5504 = 64p$
   
   $p = 86$ euros

3 a i $P(0.918) = -0.107(0.918) + 1 = 0.902$, 90.2% of sea level

ii $B(0.918) = -11.7(0.918) + 100 = 89.3^\circ C$

b There is a constant rate of change between the two variables, cooking time and boiling point temperature

c $T(B) = 15 + \frac{2}{5}(100 - B)$

d $a = 5.130$, $B(5.130) = -11.7(5.130) + 100 = 39.979^\circ C$

$T(39.979) = 15 + \frac{2}{5}(100 - 39.979) = 39$ minutes

24 minutes longer

e Domain $\{B \mid 40 \leq B \leq 100\}$, Range $\{T \mid 15 \leq T \leq 39\}$

4 a $x =$ number of portions of pepperoni, where 1 portion is 100 g

$y =$ number of portions of Parma ham, where 1 portion is 100 g

$3.5x + 6.5y = 25$

b $y = 0 \rightarrow x = \frac{25}{3.5} = 7.14$. Alfie can buy 714 g of just pepperoni

$x = 0 \rightarrow y = \frac{25}{6.5} = 3.85$. Alfie can buy 385 g of just Parma ham

c $x = y + 2$

$3.5(y + 2) + 6.5y = 25$

$10y + 7 = 25$

$y = 1.8$

$x = 3.8$

Alfie should buy 180 g of Parma ham and 380 g of pepperoni

Exercise 4G

1 a
b. i. \( f(5.7) = 4 - \frac{1}{2}(5.7) = 1.15 \), (ii) \( f(-3.2) = 2(-3.2) - 1 = -7.4 \)

c. \[ 2x - 1 = 2 \]
\[ 2x = 3 \]
\[ x = 1.5 \]
\[ 4 - \frac{1}{2}x = 2 \]
\[ \frac{1}{2}x = 2 \]
\[ x = 4 \]

d. The pieces connect at \( x = 2 \); \( 2(2) - 1 = 3 \) and \( 4 - \frac{1}{2}(2) = 3 \). Outputs match.

e. Domain \( \{x | -5 \leq x < 7\} \), Range \( \{y | -11 \leq y \leq 3\} \) since the highest point is at \( (2,3) \)

2. a. \[ f(x) = \begin{cases} 
130x & 0 \leq x < 5 \\
650 & 5 \leq x < 7 \\
130x - 260 & 7 \leq x \leq 10 
\end{cases} \]

b. \( f(5) = 130(5) = 650 \)
\( f(7) = 130(7) - 260 = 650 \)

The pieces connect so \( f \) is continuous.

c. \( f(x) = 300 \)
\[ 130x = 300 \]
\[ x = 2.31 \]
\[ f(x) = 800 \]
\[ 130x - 260 = 800 \]
\[ 130x = 1060 \]
\[ x = 8.15 \]

It took Amir \( 8.15 - 2.31 = 5.84 \) minutes

3. a. The additional cost is 6 cents per megabyte or \( 0.06 \times 1,000 \) per gigabyte = $60 per gigabyte.

\[ C(d) = \begin{cases} 
35 & 0 \leq d \leq 1 \\
35 + 60(d - 1) & d > 1 
\end{cases} \]

b. i. \( C(0.5) = 35 \)

ii. \( C(2) = 35 + 60(2 - 1) = 95 \)
The unlimited plan is better if more than 1.4 GB of data are used.

d i \[ \frac{172}{3} \times 31 \approx 1777 \text{ or approximately } 1.78 \text{ GB} \]

ii The plan charging $59 per month is cheaper, because 1.78 > 1.4.
\[ C(1.78) = 35 + 60(1.78 - 1) = 81.8 \text{. She would save }$22.80 \approx $23 \]
ii Domain: \(x \in \), Range: \(y \in\)

iii \(x = 4\)

b i

ii Domain: \(\{x \mid 0 \leq x \leq 3\}\), Range: \(\{y \mid -6 \leq y \leq 6\}\)

iii \(x = 2\)

c i

ii Domain: \(\{x \mid x \geq 0\}\), Range: \(\{y \mid y \geq -4\}\)

iii \(x = 2.5\)

2 a

b \(f^{-1}(b) = 3\)
\(f(3) = -2.5(3) + 5 = -2.5\)

c | Function | Domain | Range |
---|---|---|---|
<p>| | | |
| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th>(0 \leq x \leq 3)</th>
<th>(-2.5 \leq y \leq 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f)^{-1}</td>
<td>(-2.5 \leq x \leq 5)</td>
<td>(0 \leq y \leq 3)</td>
</tr>
</tbody>
</table>

d

\[ f(x) = f^{-1}(x) = x \]
\[ -2.5x + 5 = x \]
\[ 5 = 3.5x \]
\[ x = \frac{10}{7} \]
\[ \left( \frac{10}{7}, \frac{10}{7} \right) \]

e

\[ f(x) = f^{-1}(x) = x \]
\[ -2.5x + 5 = x \]
\[ 5 = 3.5x \]
\[ x = \frac{10}{7} \]
\[ \left( \frac{10}{7}, \frac{10}{7} \right) \]

de

\[ d(x) = 90x + 630 \]
\[ b \quad 90x + 630 = y \]
\[ 90x = y - 630 \]
\[ x = \frac{y - 630}{90} = \frac{y}{90} - 7 \]
\[ d^{-1}(x) = \frac{x}{90} - 7 \]

c i \[ d^{-1}(700) = \frac{700}{90} - 7 = 0.78 \approx 47 \text{ min} \text{ at 8:47AM} \]

ii \[ d^{-1}(880) = \frac{880}{90} - 7 = 2.78 \text{ at 10:47AM} \]

iii \[ d^{-1}(1400) = \frac{1400}{90} - 7 = 8.56 \approx 8 \text{ hours 34 min} \text{ at 4:34PM} \]

d Prague ii is most reasonable as it is closest to the middle of the day.

Exercise 4J

1 a \[ f \quad g(1) = f(4 - 1) \]
\[ = 8(3) - 25 \]
\[ = -1 \]

b \[ h \quad f(4) = h(8(4) - 25) \]
\[ = h(7) \]
\[ = \frac{3}{2}(7) \]
\[ = 10.5 \]

c \[ g \quad f(x) = g(8x - 25) \]
\[ = 4 - (8x - 25) \]
\[ = 29 - 8x \]
d  \[ h(x) = \frac{3}{2} \left( \frac{3}{2} \right)^x \]  
\[ = \frac{27}{8}x \]

2 a  
\[ c(t(x)) = 1.1x + 2 \]  
\[ t(c(x)) = 1.1(x + 2) = 1.1x + 2.2 \]

These are not equal because they are parallel linear functions with different y-intercepts.

b  
\[ t(c(x)) \text{ provides a 0.20 euro larger tip, because } 2.20 - 2.00 = 0.20 \text{ and the other parts of the two functions are equal.} \]

3 Answers can vary; samples below

a  
\[ g(x) = \frac{7}{3}x \]  \[ \text{ and } f(x) = x - 5 \]

b  
\[ g(x) = x - 2 \]  \[ \text{ and } f(x) = 4x \]

c  
\[ h(x) = 12 - 4x, \]  so one possibility is \[ g(x) = 4x \]  \[ \text{ and } f(x) = 12 - x \]

4 a  
\[ P(w(4)) = P(7(4) - 3) \]  
\[ = P(25) \]  
\[ = 50(25) - 2000 \]  
\[ = -750 \]

the company loses $750 if it operates for exactly 4 hours

b  
\[ P(w(t)) = P(7t - 3) \]  
\[ = 50(7t - 3) - 2000 \]  
\[ = 350t - 150 - 2000 \]  
\[ = 350t - 2150 \]

c  
\[ P(t) > 0 \]  
\[ 350t - 2150 > 0 \]  
\[ 350t > 2150 \]  
\[ t > 6.14 \]

The company must operate for at least 7 hours each day.

5  
\[ F(x) = C \left( \frac{9}{5}x + 32 \right) \]  
\[ = \frac{5}{9} \left( \frac{9}{5}x + 32 \right) - 17.8 \]  
\[ = x + 17.8 - 17.8 \]  
\[ = x \]  
\[ = i(x) \]

Similarly, \[ F \]  
\[ C(x) = x \]

6  
Reading from graph
\( f(x) = 0 \)
\( x = -2.5 \)

\( g(x) = -2 \)
\( x = -5 \)
\( g(x) = 5 \)
\( x = 1 \)

So \((f \circ g)(x) = 0\) when \(x = -5\) and \(x = 1\)

\textbf{Exercise 4K}

\begin{enumerate}
  \item \begin{enumerate}
    \item 41.7, 50.3, 58.9
    \item yes, \(d = 8.6\)
  \end{enumerate}

  \item \begin{enumerate}
    \item \(-83, -105, -127\)
    \item yes, \(d = -22\)
  \end{enumerate}

  \item \begin{enumerate}
    \item 151, 196, 250
    \item no, the difference is not constant, it increases by 9 each time
  \end{enumerate}

  \item \begin{enumerate}
    \item 1.25, 0.625, 0.3125
    \item no, the numbers are divided by 2 each time instead of being added or subtracted by a constant amount.
  \end{enumerate}

\end{enumerate}

2 \begin{enumerate}
  \item 9.5, 12, 14.5. This is arithmetic; \(a_1 = 9.5, \ d = 2.5\)
  \item 2650, 300, -2050. This is arithmetic; \(b_1 = 2650, \ d = -2350\)
  \item 3, 10, 21. Not arithmetic, no common difference
\end{enumerate}

3 \begin{enumerate}
  \item \(u_n = 5 + 4(n - 1)\), \(u_n = 4n + 1\)
  \item 116 = 4n + 1 has solution \(n = 28.75\). This is not a whole number, so 116 is not a term of the sequence.
4 a 95 in the second year, 105 in the third

b \( u_{10} = 85 + 10(10 - 1) = 85 + 10(9) = 175 \) employees

c \( u_n = 75 + 10n \\
75 + 10n = 285 \\
10n = 210 \\
\( n = 21 \)

21 years after opening.

5 a \( a_n = 2.6 + 1.22(n - 1) \)

b \( a_0 \rightarrow 1997 \\
a_{28} \rightarrow 2025 \\
a_{28} = 2.6 + 1.22(28 - 1) \\
= 35.54 \) m

c \( 2.6 + 1.22(n - 1) \geq 84 \\
1.22n - 1.22 \geq 81.4 \\
1.22n \geq 82.62 \\
\( n \geq 67.72 \) \\
\( n = 68 \)

In 2065.

Exercise 4L

1 a \( u_1 = -10 \\
u_2 = u_1 + 6d = 1 \\
-10 + 6d = 1 \\
6d = 11 \\
d = \frac{11}{6}

b \( u_{15} = u_1 + 14d = -10 + 14\left(\frac{11}{6}\right) = \frac{47}{3} \)

\( u_5 = u_1 + 4d \)

2 a 0 + 4d = 10 \\
d = 2.5

b \( u_3 = u_1 + 2d = 0 + 2(2.5) = 5 \)

3 a \( d = -25 \). The frog hops 25 cm closer to the finish line with each hop.

b \( u_{10} = u_1 + 9d = 975 + 9(-25) = 750 \). After 10 hops, the frog is 750 cm from the finish line.

c \( u_n = 1000 - 25n \\
1000 - 25n = 0 \\
25n = 1000 \\
\( n = 40 \)

It takes 40 hops to finish the race.
The frog hops 49 times.

4 a \[12 = u_1 + 2d\]
\[43.5 = u_1 + 9d\]

b \[31.5 = 7d\]
\[d = 4.5\]
\[12 = u_1 + 2d\]
\[12 = u_1 + 9\]
\[u_1 = 3\]

c \[u_{100} = u_1 + 99d = 3 + 99(4.5) = 448.5\]

5 a \[u_1 = 22\]
\[u_{10} = 22 + 9d = 49\]
\[9d = 27\]
\[d = 3\]
\[u_n = 22 + 3(n - 1) = 3n + 19\]

b \[3n + 19 = 106\]
\[3n = 87\]
\[n = 29\]
There are 29 rows in the theatre.

6 a \[u_1 = 12\]

b \[u_3 = 12 + 2d\]
\[u_9 = 12 + 8d\]

c \[u_9 = 2u_3\]

d \[12 + 8d = 2(12 + 2d)\]
\[12 + 8d = 24 + 4d\]
\[4d = 12\]
\[d = 3\]

e \[u_n = 12 + 3(n - 1) = 3n + 9\]
\[3n + 9 = 100\]
\[3n = 91\]
\[n = 30.3\]
So, Tyler should use an object that is at least 31 kg.

**Exercise 4M**

1 \[I = 9000 \times 0.059 \times 3\]
\[= 1593\]
Total = 9000 + 1593 = $10593
2 \[9000 = P \times 0.075 \times 7\]
\[P = \£17142.86\]

3 \[1840 = 8000 \times r \times 5\]
\[r = 0.046 = 4.6\%\]

4 To double money need $8600 interest

\[8600 = 8600 \times 0.065 \times n\]
\[n = 15.4\]
16th year

Exercise 4N

1 \[S_n = (2u_1 + (n - 1)d) \times \frac{n}{2}\]

\[S_{20} = (2(6) + (20 - 1)(-3)) \times \frac{20}{2}\]
\[= -450\]

2 \[S_n = (2u_1 + (n - 1)d) \times \frac{n}{2}\]

\[S_{30} = (2(8) + (30 - 1)(8)) \times \frac{30}{2}\]
\[= 3720\]

3 a \[u_n = 52 + 10(n - 1) = 10n + 42\]

\[10n + 42 = 462\]
\[10n = 420\]
\[n = 42\]

b \[S_n = (2u_1 + (n - 1)d) \times \frac{n}{2}\]

\[S_{42} = (2(52) + (42 - 1)(10)) \times \frac{42}{2}\]
\[= 10794\]

4 a \[1 + 5 + 9 + 13 + 17 + 21 + 25 + 29 + 33 + 37\]

b \[\sum_{i=1}^{10} (4i - 3)\]

c \[S_n = (2u_1 + (n - 1)d) \times \frac{n}{2}\]

\[S_{10} = (2(1) + (10 - 1)(4)) \times \frac{10}{2}\]
\[= 190\]

\[S_n = (2(1) + (n - 1)(4)) \times \frac{n}{2}\]
\[= (4n - 2) \times \frac{n}{2}\]
\[= n(2n - 1)\]
d \[ n(2n - 1) = 2000 \]
\[ 2n^2 - n = 2000 \]
\[ 2n^2 - n - 2000 = 0 \]
\[ n = 31.9 \]

It will take Janet 32 years to have a total of 2000 acres.

5 a \[ S_n = \frac{n}{2} (2a_1 + d(n - 1)) \]
\[ a_1 = S_1 = 4, d = -3 \]
\[ S_n = \frac{n}{2} (8 - 3(n - 1)) \text{ or } S_n = \frac{n}{2} (11 - 3n) \]

b \[ S_{10} = \frac{10}{2} (11 - 3(10)) \]
\[ = 5(11 - 30) \]
\[ = 5(-19) \]
\[ = -95 \]

c \[-250 > \frac{n}{2} (11 - 3n) \]
\[-500 > n(11 - 3n) \]
\[-500 > -3n^2 + 11n \]
\[3n^2 - 11n - 500 > 0 \]
\[n = \frac{11 \pm \sqrt{6121}}{6} \]
only positive \( n \)
\[n = 14.873 \]
\[n = 15 \]

6 a \[ u_n = 3 + 0.5(n - 1) = 0.5n + 2.5 \]
\[u_{10} = 0.5(10) + 2.5 = 7.5 \text{ km} \]

b \[ S_n = (2u_1 + (n - 1)d) \times \frac{n}{2} \]
\[S_{15} = (2(3) + (15 - 1)(0.5)) \times \frac{15}{2} \]
\[= 97.5 \text{ km} \]

7 a \[ u_1 = 22 \]
\[d = 3 \]
\[n = 29 \]
\[S_{29} = (2(22) + (29 - 1)(3)) \times \frac{29}{2} \]
\[= 1856 \]

b \[ S_n = (2u_1 + (n - 1)d) \times \frac{n}{2} \]
(2(16) + (25 - 1)d) \times \frac{25}{2} \geq 6000
12.5(24d + 32) \geq 6000
24d + 32 \geq 480
24d \geq 448
d \geq 18.7
\begin{align*}
d &= 19 \\
12.5(24(19) + 32) &= 6100
\end{align*}
\begin{align*}
d = 19 &\text{ will result in 6100 seats.}
\end{align*}

**Exercise 4O**

1. 4 + 3 + 4 + 2 \div 4 = 3.25
   
   \begin{align*}
   d &= 3.25 \\
   u_1 &= 27 \\
   u_n &= 27 + 3.25(n - 1) = 3.25n + 23.75 \\
   3.25n + 23.75 &= 1800 \\
   3.25n &= 1776.25 \\
   n &= 546.5
   \end{align*}

Tree is approximately 547 years old

2. \begin{align*}
(-0.33) + (-0.38) + (-0.36) + (-0.39) + (-0.37) + (-0.38) + (-0.36) + (-0.35) + (-0.37) \\
\end{align*} \\
\begin{align*}
&= -0.36 \\
u_n &= 2.28 - 0.36(n - 1) = 2.64 - 0.36n \\
2025 &\rightarrow u_{19} \\
u_{19} &= 2.64 - 0.36(19) = -4.2%
\end{align*}

**Exercise 4P**

1. a A linear regression is appropriate because the data displays a roughly linear trend.

![Graph](image)

b \begin{align*}
y &= 9.84 + 0.16x \\
\end{align*} using GDC

c \begin{align*}
y &= 9.84 + 0.16(120) = 29.04 \text{ euros}
\end{align*}

d 100 minutes, because it is interpolation (within the data set). Predicting for 10 minutes would be extrapolation beyond the data set.
2 a A linear regression is appropriate because the data display a roughly linear trend.

![Graph showing linear regression](image)

b \( y = 93.7x + 92.8 \)

c \( y = 93.7(1.8) + 92.8 = 261 \) kg

d A 1 cm increase in height corresponds to a 0.937 kg increase in weight.

3 a \( f_1(x) \) is graph A because it has the lower y-intercept (8.11). \( f_2(x) \) is Graph B.

<table>
<thead>
<tr>
<th>x</th>
<th>1.2</th>
<th>6.5</th>
<th>3.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>7.4</td>
<td>2.8</td>
<td>4.3</td>
</tr>
<tr>
<td>( f_1(x) )</td>
<td>7.0768</td>
<td>2.5135</td>
<td>4.9243</td>
</tr>
<tr>
<td>residual</td>
<td>-0.3232</td>
<td>-0.2865</td>
<td>0.6243</td>
</tr>
</tbody>
</table>

\[ SSR_1 = (-0.3232)^2 + (-0.2865)^2 + (0.6243)^2 = 0.576 \]

<table>
<thead>
<tr>
<th>x</th>
<th>1.2</th>
<th>6.5</th>
<th>3.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>7.4</td>
<td>2.8</td>
<td>4.3</td>
</tr>
<tr>
<td>( f_2(x) )</td>
<td>7.2044</td>
<td>2.3655</td>
<td>4.9219</td>
</tr>
<tr>
<td>residual</td>
<td>-0.1956</td>
<td>-0.4345</td>
<td>0.6219</td>
</tr>
</tbody>
</table>

\[ SSR_2 = (-0.1956)^2 + (-0.4345)^2 + (0.6219)^2 = 0.614 \]

Since \( f_1(x) \) has the smaller sum of square residuals, it is the least squares regression equation.

Exercise 4Q

1 a The correlation coefficient is weak, so prediction from a linear regression will not be accurate.

b The data does not display a linear trend; it has a curved trend. A linear regression is not appropriate.

c A linear regression is appropriate here, but Kiernan is predicting the independent variable given a value of the dependent variable. Only predictions of the dependent variable are valid.

d A linear regression is appropriate here, but Kiernan is extrapolating beyond the range of the data (\( x = 80 \) is outside the range of the data).
2. a) The graph displays a linear trend.

   ![Graph](image)

b) From GDC, \( r = 0.951 \); this is a strong positive correlation.

c) Yes, it is a linear trend with a strong correlation.

d) i) From GDC, \( T(c) = 30.3 + 1.14c \)

   ii) This is invalid, as we can use this regression to predict only the temperature from a known number of chirps.

   iii) \( T(40) = 30.3 + 1.14(40) = 76^\circ F \). Invalid, can use this regression to predict only the temperature from a known number of chirps.

e) Answers may vary. Since \( c \) is for a 15-second period, by rounding 14 seconds to 15, the Almanac’s formula becomes \( T_1(c) = c + 40 \). The gradients of the two equations are similar, and the y-intercepts differ by 10°F. \( T_1(40) = 80^\circ F \), which is 4°F different from the regression prediction.

3. a) The data follows a linear trend, as evidenced by the scatter plot:

   ![Graph](image)

b) From GDC, \( r = 0.884 \); this is a strong positive correlation

c) Yes, it is a linear trend with a strong correlation.

d) i) From GDC, \( G(b) = 342 + 0.06b \)

   ii) \( G(12320) = 342 + 0.06(12320) = \$1081.2 \)

e) A gradient of 0.06 means that for every dollar increase in Bitcoin value, graphics card prices increase by 6 cents.

Chapter Review

1. a) \( A(t) = 0 \)

   \[
   1350 - 2.7t = 0 \\
   2.7t = 1350 \\
   t = 500 \text{ hours}
   \]

b) Domain: \( \{t \mid 0 \leq t \leq 500\} \)

Range: \( \{A \mid 0 \leq A \leq 1350\} \)
c

\[ A(t) = 1350 - 2.7t = 1336.5; \] after 5 hours, there are 1336.5 m\(^2\) of surface area remaining to be cleaned.

2 a function; not linear

b function; linear

c not a function: two cylinders with the same volume but different heights will have different radii.

3 a \(x\) = babies per woman and \(y\) = life expectancy. Equation from GDC: \(y = -5.89x + 88.2\)

b The scatter plot shows a linear relationship between the two variables; the correlation coefficient, \(r = -0.726\), shows at least a moderately strong negative correlation.

c When \(x = 2\), \(y = 88.2 - 5.89(2) = 76.4\) years. This is a valid prediction because it is interpolation.

d Gradient of \(-5.89\) in the regression equation means that an increase of 1 baby per woman corresponds to a decrease of 5.89 years of life expectancy.

4 a Graph 1, because it has a y-intercept of 0

b Graph 1: \(\frac{150}{1800} = \frac{1}{12}\); the plane uses 12 litres to travel 1 km

\[ \text{Graph 2: } \frac{260 - 20}{3000 - 0} = \frac{240}{3000} = 0.08; \text{ the plane takes 8 minutes to travel 100 km} \]

c \(T(d(x)) = 20 + 0.08 \left(\frac{1}{12}x\right) = 20 + 0.0067x\)

\[ T(150000) = 20 + 0.0067(150000) = 1025 \text{ minutes or 17 hours 5 minutes} \]

5 a \(u_0 = 400\)

\(d = 150\)

\[ u_k = 550 + 150(n - 1) = 150n + 400 \]

150n + 400 = 9500

150n = 9100

\(n = 60.67\)

61 months

b \(400 + 12(150) = $2200\)

9500 - 2200 = $7300 remaining

\(0 + k + 2k + 3k + 4k + \ldots + 11k = 7300\)

\(0 + 11k \left(\frac{12}{2}\right) = 7300\)

66k = 7300

\(k = 110.61\)
\[ k = \$111 \]

\[ c \quad 400 + 150n + \left( \frac{n}{2} \right)(2(0) + 111(n - 1)) = 20000 \]
\[ 800 + 300n + 111n^2 - 111n = 40000 \]
\[ 111n^2 + 189n - 39200 = 0 \]
\[ n = 17.96 \]

18 months in total, so 6 more months.

6 a \[ C(d) = 0.06d + 49 \]

<table>
<thead>
<tr>
<th>b i</th>
<th>Destination</th>
<th>Cost to drive</th>
<th>Cost to fly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brussels</td>
<td>( C(420) = 49 + 0.06(420) = 74.20 )</td>
<td>130</td>
<td></td>
</tr>
<tr>
<td>Hamburg</td>
<td>( C(940) = 49 + 0.06(940) = 105.40 )</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>Paris</td>
<td>( C(1030) = 49 + 0.06(1030) = 110.80 )</td>
<td>240</td>
<td></td>
</tr>
</tbody>
</table>

Brussels: drive (costs €74.20); Hamburg: fly (drive costs €105.40); Paris: drive (costs €110.80)

\[ ii \quad 74.20 + 150 + 110.80 = €335 \]

7 \[ I = P \times r \times n \]
\[ P + I = P + 8\%P \]
\[ I = 0.08P \]
\[ 0.08P = P \times r \times 5 \]
\[ 0.08 = 5r \]
\[ r = 0.016 \]
\[ 5 \times 0.016 \times P = 3200 \]
\[ P = 40000 \]

Investment = €40000

Rate = 1.6%

8 a \[ 0.15 \times 46605 + 0.205 \times (93208 - 46605) + 0.26 \times (122000 - 93208) = 24030.29 \]

b \[ 0.15 \times 46605 = 6990.75 \]
\[ 6990.75 + 0.205 \times (93208 - 46605) = 16544.37 \]
\[ 16544.37 + 0.26 \times (144489 - 93208) = 29877.43 \]
\[ 29877.43 + 0.29 \times (205842 - 144489) = 47669.80 \]

\[ T(x) = \begin{cases} 
0.15x & 0 \leq x \leq 46605 \\
6990.75 + 0.205(x - 46605) & 46605 < x \leq 93208 \\
16544.37 + 0.26(x - 93208) & 93208 < x \leq 144489 \\
29877.43 + 0.29(x - 144489) & 144489 < x \leq 205842 \\
47669.80 + 0.33(x - 205842) & x > 205842 
\end{cases} \]

c i One payment: \( T(162000) + T(122000) = 29877.43 + 0.29(162000 - 144489) + 24030.29 = 58985.91 \)

Two payments: \( 2 \times T(142000) = 2 \times (16544.37 + 0.26(142000 - 93208)) = 58460.58 \)

ii Ian should choose two payments; he will save CA$525.33.
9 a  \( f^{-1}(x) = \begin{cases} 15 - 3x, & 4 \leq x \leq 7 \\ \frac{1}{2}x + 5, & x < 4 \end{cases} \)

Domain: \( \{x \mid x \leq 7\} \)
Range: \( \{y \mid y \geq -6\} \)

b Inverse function would be \( f^{-1}(x) = \begin{cases} 15 - 3x, & 4 \leq x \leq 7 \\ \frac{1}{2}x + 1, & x > 4 \end{cases} \)

But for \( 4 \leq x \leq 7 \), there are two possible values of \( f^{-1}(x) \) and so it is not a function.

In later chapters you will learn about the horizontal line test which is a simple way to determine whether or not a function has an inverse.

Exam Style Questions

10 a i \( m = \frac{140 - 100}{100 - 70} = \frac{4}{3} \)

ii \( c = y - mx = 100 - \frac{4}{3}(70) = 6 \frac{2}{3} \)

b \( m \) is positive, so correlation is positive

c \( \bar{y} = \frac{4}{3}(90) + 6 \frac{2}{3} = 127 \)

d \( y = \frac{4}{3}(60) + 6 \frac{2}{3} = 86.7 \)

11 a \( 70 + 8.35d < 30 + 12.5d \)
\( 40 < 4.15d \)
\( 9.64 < d \)

Car-nage is cheaper for 10 days or more. Therefore, Abel’s holiday is at least 10 days.

b \( 14 \leq d \leq 21 \)
\( 70 + 8.35(14) = 186.90 \)
\( 70 + 8.35(21) = 245.35 \)
\( \{C \mid 186.90 \leq C \leq 245.35\} \)

12 a i From GDC, \( r = 0.849 \)

ii Strong positive

iii From GDC, \( y = 0.24 + 0.94x \)

b i From GDC, \( r = 0.267 \)

ii Weak positive

iii Data does not show a strong enough linear correlation for a linear regression line to be valid.
13 a  \[ b = \frac{0-5}{12-0} = \frac{-5}{12} \]
\[ a = 5 \]

b  \[ y = 5 - \frac{5}{12}x \]
\[ \frac{5}{12}x = 5 - y \]
\[ x = 12 - \frac{12}{5}y \]
\[ f^{-1}(x) = 12 - \frac{12}{5}x \]

c  \[ 5 - \frac{5}{12}x = 12 - \frac{12}{5}x \]
\[ \frac{119}{60}x = 7 \]
\[ x = \frac{60}{17} = \frac{39}{17} \]

d  \( h^1(x) \) is a reflection of \( h(x) \) on the line \( y = x \) so they will intersect when \( h(x) = x \).

14 a  \( (3a + b) - 7 = (5a - 6b) - (3a + b) = (2a + 9b + 4) - (5a - 6b) \)
\[ a + 8b = 7 \]
\[ 6a - 14b = 11 \]
\[ a = 3 \]
\[ b = 0.5 \]

b  Substituting values gives the sequence 7, 9.5, 12, 14.5, ...
\[ u_1 = 7 \text{ and } d = 2.5 \]
\[ S_n = (2u_1 + (n-1)d) \times \frac{n}{2} > 1000 \]
\[ (2(7) + 2.5(n-1)) \times \frac{n}{2} > 1000 \]
\[ n(11.5 + 2.5n) > 2000 \]
\[ 2.5n^2 + 11.5n - 2000 > 0 \]
\[ n > 26.0776... \]
\[ n = 27 \]

15 a  \( \{ f | f \in \} \quad f(x) \in \)

b  \( (g \circ f)(x) \geq 18 \)

c  \( (f \circ g)(x) = 2x^2 + 18 - 24 = 2x^2 - 6 \)
\[ 2x^2 - 6 = 0 \]
\[ x = \pm 1.73 \quad (or \quad \pm \sqrt{3}) \]
\[ f^{-1}(x) = x + 24 \]
\[ (g \circ f \circ f^{-1})(x) = g(x) \]
\[ (g \circ f)(x + 24) = 2(x + 24)^2 + 18 \]
\[ = 2x^2 + 96x + 1170 \]
\[ g(x) = 2x^2 + 96x + 1170 \]