7 Modelling rates of change: exponential and logarithmic functions

Skills check

1 a \((x^2)^3 = x^{2\cdot3} = x^6\)

b \(\left(\frac{a}{b}\right)^{-1} = \frac{a^{-1}}{b^{-1}} = \frac{b}{a}\)

c \(x^2\sqrt{x} = x^2 \cdot x^{\frac{1}{2}} = x^{\frac{3}{2}} = \frac{1}{\sqrt{x^3}}\)

d \(\sqrt[3]{x} = x^{\frac{1}{3}}\sqrt{x} = x^{\frac{4}{3}} = \sqrt[3]{x^4}\)

e \(\left(\sqrt[5]{x}\right)^5 = x^{\frac{5}{5}} = x = \sqrt[5]{x^5}\)

2 a \(0.03 \times 24 = 0.72\)

b \(0.15 \times 72 = 10.8\)

c \(0.28 \times 150 = 42\)

3 a

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
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<tbody>
<tr>
<td>y</td>
<td>4</td>
<td>1</td>
<td>-2</td>
<td>-5</td>
</tr>
</tbody>
</table>

b \(f(12) = 1 - (3 \times 12) = -35\)

c

4 a \(\sum_{i=1}^{10} (2i + 1) = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 = 120\)

b \(\sum_{i=5}^{10} (2i + 1) = 11 + 13 + 15 + 17 + 19 + 21 = 96\)

c \(\sum_{i=1}^{4} i^3 = 1 + 8 + 27 + 64 = 100\)
Exercise 7A

1 a \( r = \frac{10}{5} = 2 \), \( u_2 = 5 \times 2^0 = 320 \), \( u_n = 5 \times 2^{n-1} \)

b \( r = \frac{15}{-3} = -5 \), \( u_6 = -3 \times (-5)^6 = 234375 \), \( u_n = -3 \times (-5)^{n-1} \)

c \( r = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}} \), \( u_6 = \sqrt{2} \times \sqrt{3} = \sqrt{2} \times 9 \times \sqrt{3} \approx 22.0 \), \( u_n = \sqrt{2} \times \sqrt{3}^{n-1} \)

d \( r = \frac{1}{3/2} = \frac{2}{3} \), \( u_4 = \frac{3}{2} \times \left( \frac{2}{3} \right)^3 = \frac{4}{9} \), \( u_n = \frac{3}{2} \left( \frac{2}{3} \right)^{n-1} = \left( \frac{2}{3} \right)^{n-2} \)

e \( r = \frac{20}{2} = 10 \), \( u_{10} = 2 \times (10)^9 = 2 \times 10^9 \), \( u_n = 2 \times 10^{n-1} \)

2 a \( u_4 = 24 = 3 \), \( r^3 \Rightarrow r = \sqrt[3]{8} = 2 \). Hence \( u_n = 3 \times 2^{n-1} \)

b \( u_6 = 32r^5 = 243 \Rightarrow r = \frac{\sqrt[5]{243}}{32} = 1.5 \). Hence \( u_n = 32 \times 1.5^{n-1} \)

c \( u_5 = 81 = 1r^4 \Rightarrow r = \sqrt[4]{81} = 3 \). Hence \( u_n = 3^n \)

d \( u_4 = \frac{1}{2}r^4 = \frac{4}{27} \Rightarrow r = \sqrt[4]{\frac{8}{27}} = \frac{2}{3} \). Hence \( u_n = \left( \frac{2}{3} \right)^{n-1} \)

e \( u_4 = -1458 = -2r^6 \Rightarrow r = \sqrt[6]{729} = 3 \). To give positive and negative terms, \( r = -3 \). Hence \( u_n = -2(-3)^{n-1} \)

f \( u_3 = 13.5 = u_1r^2 \) and \( u_6 = 364.5 = u_1r^5 \) hence \( \frac{u_6}{u_3} = r^3 = 27 \Rightarrow r = 3 \). Then

\( u_3 = 13.5 = u_1 \times 3^2 \Rightarrow u_1 = \frac{13.5}{9} = 1.5 \) which gives \( u_n = 1.5 \left( 3 \right)^{n-1} \).

3 a \( \frac{u_1}{u_2} = \frac{10}{5} = 2 = \frac{u_2}{u_3} \) hence, it’s a geometric sequence.

b \( u_7 = 2 \times 5^6 = 31250 \)

c \( u_{30} = 2 \times 5^{29} = 3.73 \times 10^{30} \). This is obviously not a reasonable answer. The total population of the world is less than \( 10^{10} \) people.

Exercise 7B

1 a As the sequence is geometric, then \( P_{2018} = P_{2016} r^2 \Rightarrow r = \sqrt[2]{\frac{264500}{200000}} = 1.15 \). Hence,

\( P_{2017} = 200000 \times 1.15 = 230000 \)

b \( P_{2020} = 200000 \times 1.15^4 = 349801 \)

c No. It’s too fast. In just 4 years the population almost doubled!
2  \( P_{2019} = 2.2 \times (1 + 0.0265)^4 = 2.44 \)

3  \( P_6 = P_0 (1 - p)^6 = 45000 \times (1 - 0.05)^6 = €33079 \)

4  a  \( P_3 = 15000 \times (1 - 0.12)^3 = €10222 \)

  b  \( €5000 = 15000 \times (0.88)^y \) from the GDC  \( y = 8.59 \). Hence, it would take 8.59 years.

5  a  \( r = 1 + 0.125 = 1.125 \)

  b  \( r = 1 - 0.073 = 0.927 \)

  c  \( r = 0.89 \)

  d  \( r = 1 + 0.001 = 1.001 \)

6  Here we consider the initial year as the year 0.

  a  \( P_7 = 1.2 \times 1.05^7 = 1.69 \text{ m}, \) % change is applied 7 times

  b  The 7th measurement is 6 years later, so \( P_6 = 1.2 \times 1.05^6 = 1.61 \text{ m}, \) % change is applied 6 times

  c  We can say that 8 years have passed, so \( P_8 = 1.2 \times 1.05^8 = 1.77 \text{ m}, \) % change is applied 8 times

7  a  \( P_{2019} = 7.7 \times 1.0172^{10} = 9.13 \text{ billion people} \)

  b  \( P_{2019} = 7.7 = P_{2012} \times 1.0172^7 \Rightarrow P_{2012} = 7.7 \times 1.0172^{-7} = 6.83 \text{ billion people.} \)

  c  We treat the end of 2040 the same as the beginning of 2041. Hence, \( P_{2041} = 7.7 \times 1.0172^{22} = 11.21 \text{ billion people.} \)

  d  \( 2 \times 7.7 = 7.7 \times 1.0172^r \) years. From the GDC  \( y = 40.64 \approx 41 \), so population would have doubled by the beginning 2060, i.e. during 2059.

  e  2090 is 71 years from 2019, so \( 10 = 7.7 \times (1 + r)^{71} \Rightarrow r = \left(\frac{10}{7.7}\right)^{1/71} - 1 = 0.003688 \). The annual rate should be 0.37%.

**Exercise 7C**

1  a  \( u_7 = 1 \times 1.4^6 = 7.53 \text{ cm} \)

  b  The total material is just the sum of the parts, so \( S_7 = 1 \times 1.4^7 - 1 = 23.85 \text{ cm} \) will be used.

2  a  Here, treating “millions” as the units, \( u_1 = 12 \) and \( r = 1.1 \), therefore \( u_6 = u_1 r^5 = 12 \times 1.1^5 \approx 19.3 \). That is, 19 million were created in the sixth hour.

  b  \( S_6 = \frac{u_i (r^6 - 1)}{r - 1} \frac{12(1.1^6 - 1)}{(1.1 - 1)} = 92.6 \text{ million} \)
Worked solutions

3 a \[ r = 1 + 0.062 = 1.062 \]

b Treating “billion” as the units, \( E_{2019} = 1.7 \times 1.062^{11} = 3.29 \) billion

c \[ \%_{2008-2019} = \frac{3.29 - 1.7}{1.7} \times 100\% = 94\% \]

d We look for the number \( y \) of years such that
\[ 1.7 \times 1.062^y > 3. \] From the GDC \( y = 9.44 \) years This will happen after 10 years in 2018.

e Soon, almost everyone in the world will have an email address and the number of new email users will not grow as fast.

4 a We look for \( S_{10} = 33 \times S_5 = 3 \times \frac{r^5 - 1}{r - 1} = 33 \times 3 \times \frac{r^5 - 1}{r - 1} \Rightarrow r^{10} = 33(r^5 - 1) + 1 \)
\[ \Rightarrow r^{10} - 33r^5 + 32 = 0. \] This can be solved directly on the GDC or note that we can transform this into a quadratic equation \( (r^5)^2 - 33r^5 + 32 = 0, \) so
\[ r^5 = \frac{33 \pm \sqrt{33^2 - 4 \times 32}}{2} = 32 \text{ or } 1. \] As the sequence is geometric then \( r \neq 1, \) so \( r = \frac{3}{2} \)

b We look for \( u_n = 3 \times 2^{n-1} < 1000, \) From the GDC \( n < 9.38 \) so \( u_9 = 3 \times 2^9 = 768 \) is the last term below 1000.

c \[ S_n = 3 \times \frac{2^n - 1}{2 - 1} > 1000 \quad n > 8.39. \] So \( S_9 = 3 \times \frac{2^9 - 1}{2 - 1} = 1533 \) is the lowest sum greater than 1000.

d \[ S_k = 33825S_5, \text{ where } S_5 = 3 \times \frac{2^5 - 1}{2 - 1} = 93, \text{ so } \]
\[ S_k = 33825 \times 93 \text{ from the GDC } k = 20 \]

5 a We get the ratio \( r = \frac{10}{2} = 5. \) So \( u_k = 2 \times 5^{k-1} = 781 \text{ 250} \) from the GDC \( k = 9 \)

Hence, the sum up to \( u_k \) is \( S_9 = 2 \times \frac{5^9 - 1}{5 - 1} = 976 \text{ 562}. \)

b \[ r = \frac{9.6}{6.4} = 1.5 \Rightarrow u_k = 6.4 \times 1.5^{k-1} = 164.025. \] From the GDC \( k = 9 \)
Hence, the sum up to \( u_k \) is \( S_k = 6.4 \times \frac{1.5^0 - 1}{1.5 - 1} = 479.275 \).

\[ c \quad u_n = \frac{8}{3} \left( \frac{1}{4} \right)^{n-1} = \frac{1}{6144} \]. From the GDC \( n = 8 \)

Hence, the sum is \( S_8 = \frac{8}{3} \times \frac{\frac{1}{4} - 1}{\frac{1}{4} - 1} = \frac{21845}{6144} = 3.56 \).

6 \ a \quad r = 1 - 0.002 = 0.998 

\[ b \quad t_5 = 30.4 \times 0.998^4 = 30.2 \text{ seconds} \]

\[ c \quad \text{After 20 days we expect her to do } t_{20} = 30.4 \times 0.998^{19} = 29.27 < 29.3 \text{ seconds. She will make it that day!} \]

Exercise 7D

1 \ a \quad r = \frac{0.002}{0.001} = 2 > 1 . \text{ Hence, the series diverges.} 

\[ b \quad r = \frac{500 000}{1 000 000} = \frac{1}{2} < 1 \Rightarrow S_n = \frac{1 000 000}{1 - \frac{1}{2}} = 2 000 000 \]

\[ c \quad r = -\frac{3}{1} = -3 , \quad |r| > 1 . \text{ Hence, the series diverges.} \]

\[ d \quad r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} < 1 \Rightarrow S_n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \]

2 \ a \quad S_\infty = 7u_1 \Rightarrow \frac{u_1}{1-r} = 7u_1 \Rightarrow r = 1 - \frac{1}{7} \Rightarrow r = \frac{6}{7} 

\[ b \quad \frac{S_k}{\frac{1}{2}} > \frac{S_\infty}{\frac{1}{2}} \]

\[ \frac{u_1 \left( 1 - \left( \frac{6}{7} \right)^k \right)}{1 - \frac{6}{7}} > \frac{1}{2} u_1 \]

\[ 1 - \left( \frac{6}{7} \right)^k > \frac{1}{2} \]

\[ \left( \frac{6}{7} \right)^k < \frac{1}{2} \]

\[ k > 4.50 \]

So, the sum of the first 5 terms is the lowest sum that is more than half of \( S_\infty \).
3  a  \[ S_\infty = \frac{4}{1-r} = 24 \Rightarrow r = \frac{5}{6} \]

b  \[ u_1 = 4, u_2 = 4 \times \frac{5}{6} = \frac{10}{3}, u_3 = 4 \times \left(\frac{5}{6}\right)^2 = \frac{25}{9} \]

c  \[ v_1 = 16, \quad v_2 = \frac{100}{9}, \quad v_3 = \frac{625}{81} \]

d  \[ r_{\text{new}} = \frac{100}{9 \times 16} = \frac{25}{36} = \left(\frac{5}{6}\right)^2 = r^2 \]

e  \[ S_{n(\text{new})} = \frac{16}{1 - \frac{25}{36}} = \frac{576}{11} \]

f  \[ \frac{S_{n(\text{new})}}{S_n^2} = \frac{\frac{576}{11}}{24^2} = \frac{1}{11}. \text{ The series of squared terms is much less than the squared series of original terms.} \]

4  a  \[ B_1 = 2 \times 0.75 = 1.5 \text{ m} \]

b  \[ B_2 = 1.5 \times 0.75 = 1.125 \text{ m} \]

c  \[ B_3 = 1.125 \times 0.75 = 0.84375 \text{ m} \]

d  \[ r = 0.75 \]

e  \[ \text{No, because } r \times (0.75)^k \neq 0 \text{ for any finite } k > 0. \]

f  \[ S_n = \frac{2}{1-0.75} = 8 \text{ m} \]

5  a  \[ a_1 = \frac{\pi 4^2}{2} = 8\pi \text{ cm}^2 \]

b  \[ a_2 = \frac{\pi 4^2}{4} = 4\pi \text{ cm}^2 \]

c  \[ S_2 = 8\pi + 4\pi = 12\pi \text{ cm}^2 \]

d  \[ \text{The area is halved each time, hence } r = 1/2. \text{ Thus, } a_5 = 8\pi \times \left(\frac{1}{2}\right)^4 = \frac{\pi}{2} \text{ cm}^2 \]

e  \[ S_5 = 8\pi \left(\frac{1}{2}\right)^5 - 1 \left(\frac{1}{2} - 1\right) = 15.5\pi \text{ cm}^2 \]

f
\[ S_k > 0.99 \times 16\pi \]
\[ 0.99 \times 16\pi < 8\pi \left( \frac{1}{2} \right)^k \]
\[ k > 6.64 \]

Hence, 7 students are needed to cover at least 99% of the circle.

g  \[ S_n = \frac{8\pi}{1 - \left( \frac{1}{2} \right)^k} = 16\pi \text{ cm}^2 \], which is the total area of the circle.

6 a  It's geometric, so \( r = 0.89 \), thus \( u_3 = 1 \times 0.89^2 = 0.792 \) m

b  \[ u_k = 1 \times 0.89^{k-1} < 0.5 \]
\[ k > 6.9 \]

After 7 oscillations, he will need to push the swing again.

c  \[ S_7 = \frac{0.89^7 - 1}{0.89 - 1} = 5.07 \text{ m} \]

d  \[ S_n = \frac{1}{1 - 0.89} = 9.09 \text{ m} \]

Exercise 7E

Note: When using the Finance app if there are no payments into the account beyond the initial payment then set PMT as 0 and P/Y as 1. In this case N will give the answer as the number of years. If the P/Y is set as other than 1 the value of N on most calculators will be the number of payments rather than the number of years.

1  In 15 years:
\[ FV_{\text{Oswald}} = 5000 \left( 1 + 0.037 \right)^{15} = \€8622.86 \]
\[ FV_{\text{Martha}} = 5000 \left( 1 + \frac{0.035}{12}\right)^{12 \times 15} = \€8445.84 \]

Oswald will have more.

2 a  \( FV = 50000 \left( 1 + 0.032 \right)^{10} = \€68512.05 \text{ NIS} \)

or

\[
\begin{array}{l}
N=10 \\
I\%=3.2 \\
PV=-50000 \\
PMT=0 \\
FV=68512.05232 \\
P/Y=1 \\
C/Y=1 \\
\end{array}
\]
b
- N=22.00560358
- I%=3.2
- PV=-50000
- PMT=0
- FV=100000
- P/Y=1
- C/Y=1

22 years.

c
- N=20
- I%=3.526492384
- PV=-50000
- PMT=0
- FV=100000
- P/Y=1
- C/Y=1

This corresponds to a nominal annual interest of 3.53%.

3 a
- N=5
- I%=5.099998897
- PV=-4500
- PMT=0
- FV=5803.94
- P/Y=1
- C/Y=12

She has an annual interest rate of 5.1% compounded monthly.

b A 50% increase means for her to have $4500 + 4500 \times 0.5 = £6750$. Thus,

- N=7.967180396
- I%=5.099998897
- PV=-4500
- PMT=0
- FV=6750
- P/Y=1
- C/Y=12

She would need 8 years to get a 50% increase in the amount invested.

4 Compounded quarterly means that C/Y is equal to 4,

- N=5.456727663
- I%=7.5
- PV=-1000
- PMT=0
- FV=1500
- P/Y=1
- C/Y=4

He will be able to buy it after 6 years.

5 $6762.56 = PV \left(1 + \frac{0.012}{2}\right)^{2 \times 10} \Rightarrow PV = \frac{6762.56}{1.006^{20}} = €6000$
6  a  \( k = 4 \), In one year the rate would be 
\[
1 + \frac{0.06}{4} = 1.0614 .
\]
Hence, the effective yearly interest rate is of 6.14%.

This can also be calculated using the Finance app. Set PV as -100 and N as 1 year. The final value will give the amount the 100 has increased by from which the effective interest can be calculated.

\[
\begin{align*}
N &= 1 \\
I &= 6 \\
PV &= -100 \\
PMT &= 0 \\
\text{FV} &= 106.1363551 \\
P/Y &= 1 \\
C/Y &= 4
\end{align*}
\]

100 has increased to 106.14 hence a 6.14% increase.

b  \( k = 12 \), In one year the rate would be 
\[
1 + \frac{0.06}{12} = 1.0617 .
\]
Hence, the effective yearly interest rate is of 6.17%.

or

\[
\begin{align*}
N &= 1 \\
I &= 6 \\
PV &= -100 \\
PMT &= 0 \\
\text{FV} &= 106.1677812 \\
P/Y &= 1 \\
C/Y &= 12
\end{align*}
\]

7  a  Adjusting for inflation, \( r \Rightarrow r - i = 0.3\%

b  \( FV = 20000 \left( 1 + \frac{0.003}{12} \right) = 20060 \)

8  a  Adjusting for inflation \( r \Rightarrow r - i = 0.6\%

b  \( FV = 2000 \left( 1 + 0.006 \right) = 2012 \)

or

\[
\begin{align*}
N &= 1 \\
I &= 0.6 \\
PV &= -2000 \\
PMT &= 0 \\
\text{FV} &= 2012 \\
P/Y &= 1 \\
C/Y &= 1
\end{align*}
\]

Exercise 7F

1  a  N is the number of payments and the payments are made monthly. In 10 years there are 120 months so N=120
The payment per month is $111.02

b The balance can be found either by changing N to 60 (for 5 years) or by using the balance function in your finance app. If using the full value obtained for the payment in part a (111.0205...), the solution is $5742.60, but as the payments can only be for whole number of cents $11.02 should be used to give $5742.63.

Both solutions would be acceptable in an examination.

2 a \( N = 6 \times 12 = 72 \)

\[
\begin{array}{l}
N=72 \\
I\%=6.3 \\
PV=40000 \\
PMT=-667.9725942 \\
FV=0 \\
P/Y=12 \\
C/Y=4 \\
\end{array}
\]

Payments are 667.97AED per month

b At the end of three years \( N = 3 \times 12 = 36 \) The final value with give the balance which is equal to 21869.79AED

3 a Payments are monthly so N is the number of months: \( N = 25 \times 12 = 300 \)

\[
\begin{array}{l}
N=300 \\
I\%=8 \\
PV=0 \\
PMT=1000 \\
FV=957366.5705 \\
P/Y=12 \\
C/Y=12 \\
PMT:END \text{ BEGIN} \\
\end{array}
\]

The value of the investment after 25 years is TRY 957366.57

b Interest per month is \( 951026.39 \times \frac{8}{12 \times 100} = 6340.17 \), as \( 6340.17 > 1200 \) the money gained by the account each month is much greater than the money being paid out so the money will never run out.

4 a
Payment is €178.13 per month

b \( N = 12 \times 5 = 60 \)

Money needing to be paid off is €2078.61

If payment is adjusted to €178.13 which would be the actual amount paid the amount owing would be €2078.29

5

The monthly repayment amount is $692.61

As he can pay $700, then he can afford to buy the car.

Exercise 7G

1 a \( f(x) = 4^x + 1 \)

i Crosses \( y \)-axis at \((0,1 + 1) = (0,2)\)

ii Horizontal asymptote at \( y = 1 \)

iii Increasing as \( 4 > 1 \)

b \( f(x) = 0.2^x - 3 \)

i Crosses \( y \)-axis at \((0,1 - 3) = (0,-2)\)
ii Horizontal asymptote at $y = -3$

iii Decreasing as $0.2 < 1$

c $f(x) = 5^x$

i Crosses $y$-axis at $(0,1)$

ii Horizontal asymptote at $y = 0$

iii Increasing as $5 > 1$

d $f(x) = 3^{0.1x} + 2$

i Crosses $y$-axis at $(0,1 + 2) = (0,3)$

ii Horizontal asymptote at $y = 2$

iii Increasing as $3 > 1$

e $f(x) = 3 \times 2^x - 5$

i Crosses $y$-axis at $(0,3 - 5) = (0,-2)$

ii Horizontal asymptote at $y = -5$

iii Increasing as $2 > 1$

f $f(x) = 4 \times 0.3^2x + 3$

i Crosses $y$-axis at $(0,4 + 3) = (0,7)$

ii Horizontal asymptote at $y = 3$

iii Decreasing as $0.3 < 1$

g $f(x) = 5 \times 2^{0.5x} - 1$

i Crosses $y$-axis at $(0,5 - 1) = (0,4)$

ii Horizontal asymptote at $y = -1$

iii Increasing as $2 > 1$

h $f(x) = 2 \times 2.5^{x-1} = 2 \times \left(\frac{1}{2.5}\right)^x - 1$

i Crosses $y$-axis at $(0,2 - 1) = (0,1)$

ii Horizontal asymptote at $y = -1$

iii Decreasing as $\frac{1}{2.5} < 1$

2 a $2 \times 4^x + 5 = 2(4^x) + 5 = 2 \times 16^x + 5$

b $7 \times 0.5^{3x} + 2 = 7 \left(\frac{1}{2}\right)^{3x} + 2 = 7 \times 8^x + 2$
3 a  \( S(0) = 12 + 10 \times 1 = 22 \)

b  We want that \( S(t) = 15 = 12 + 10 \times 1.2^{-t} \) \( \rightarrow t = 6.6 \) hours from the GDC.

c  It’s a translation of \( \left( \frac{0}{2} \right) \). Thus, \( S_1(t) = 14 + 10 \times 1.2^{-t} \)

d  It’s a stretch parallel to the \( S \) axis of \( 2 \). Thus \( S_2(t) = 12 + 10 \times 2^{-t} \)

e  It’s a stretch parallel to the \( t \) axis of \( 2 \). Thus \( S_3(t) = 24 + 20 \times t^{-t} \)

Exercise 7H

<table>
<thead>
<tr>
<th></th>
<th>y-intercept</th>
<th>Horizontal asymptote</th>
<th>Growth or decay</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( f(x) = e^x + 3 )</td>
<td>(0,4)</td>
<td>( y = 3 )</td>
<td>Growth</td>
</tr>
<tr>
<td></td>
<td>( f(x) = 2e^{-x} + 4 )</td>
<td>(0,6)</td>
<td>( y = 4 )</td>
<td>Decay</td>
</tr>
<tr>
<td></td>
<td>( f(x) = 0.2e^{0.3x} - 2 )</td>
<td>(0,-1.8)</td>
<td>( y = -2 )</td>
<td>Growth</td>
</tr>
<tr>
<td></td>
<td>( f(x) = 5 - 2e^{-3x} )</td>
<td>(0,3)</td>
<td>( y = 5 )</td>
<td>Decay</td>
</tr>
</tbody>
</table>

2 a  

b  

c  

d  

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3 \( T(t) = 24.5 + 55.9e^{-0.0269t} \)

a Crosses T-axis at \((0, 80.4)\)

b The y-intercept is the temperature of the water when it is initially poured into the cup.

c It’s decay as the coefficient of the exponent is negative.

d The temperature decays over time as the water cools.

e \( y = 24.5 \)

f The horizontal asymptote represents room temperature. After a long time has passed, the water will be in equilibrium with the room.

g \( 24.5 < T \leq 80.4 \)

The upper bound is the temperature of the water when it was initially poured into the cup and the lower bound is the equilibrium temperature after a long time.

h

4 a The piecewise function must have the same value for \( x = 3 \) on both sides to be continuous. Thus, \( a \times 3^3 = 2e^{3} \Rightarrow a = 2\left(\frac{e^3}{3}\right) = 1.488. \)

b The piecewise function must have the same value for \( x = 4 \) on both sides to be continuous. Thus, \( a \times 4^2 + 8 = 2e^{4} \Rightarrow a = \frac{24}{16} = \frac{3}{2}. \)

Exercise 7I

1 a \( n(23) = 252 \Rightarrow 252 = 8e^{6k} \Rightarrow k = 0.15 \)

b One hour is 60 minutes, so \( n(60) = 8e^{0.15 \cdot 60} = 64825 \) bacteria.

c \( n(T) > 100 000 \Rightarrow 8 \times e^{0.15\cdot T} > 100 000 \) minutes.

Using the table (or graph / solver ) function in the calculator we find that in 63 minutes there will be over 100 000 bacteria.

2 a Initial mass when \( t = 0 \). Thus, \( m(0) = 952 \) kg.

b \( m(2.5) = 952e^{\frac{2.5}{3}} = 414 \) kg.

c \( m(t_{\text{half}}) = \frac{1}{2} \times 952 \Rightarrow e^{\frac{t_{\text{half}}}{3}} = \frac{1}{2} \Rightarrow t_{\text{half}} = 2.08 \) hours from the GDC.

d \( m(t_{10\%}) = 0.1 \times 952 \Rightarrow t_{10\%} = 6.91 \) hours.
3  a  \( y = -3 \)

b

\[ y = -3 \]

The domain of the inverse function is the range of the original function and vice versa. Hence domain is \( x > -3 \) and the range is \( f^{-1}(x) \in \mathbb{R} \).

4  a  As initially there were 5000 ml, then \( a = 5000 \).

b  \( V \left( \frac{35}{60} \right) = 151 = 5000e^{b \cdot 35} \Rightarrow b = -0.100 \)

\[ V(\frac{35}{60}) = 151 = 5000e^{b \cdot 35} \Rightarrow b = -0.100 \]

The value outside the equation is 151.

\[ b = -0.100 \]

\[ V(T) < 100 \Rightarrow 5000e^{0.10T} < 100 \] From the table function (or graph / solver) on the calculator \( T \geq 40 \) so the least value of \( T \) is 40 minutes.

5  a

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value (CZK) millions</td>
<td>7.17</td>
<td>7.18</td>
<td>7.21</td>
<td>7.22</td>
<td>7.24</td>
<td>7.27</td>
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</table>

<table>
<thead>
<tr>
<th>Month</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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</thead>
<tbody>
<tr>
<td>Value (CZK) millions</td>
<td>7.27</td>
<td>7.33</td>
<td>7.35</td>
<td>7.38</td>
<td>7.44</td>
<td>7.52</td>
</tr>
</tbody>
</table>

b  It’s an increasing set of points with increasing differences, possibly approaching an asymptote at \( v=0 \)

c  Using calculator, \( v = 0.144(1.11)^t \)

d  Using the calculator \( R^2 = 0.998 \)
e. \( v = 0.150(1.10)^t > 2 \implies t > 25.2 \), so 26 months or about 2 years and 2 months.

6 a

b HPC and the rate of change are constantly increasing.

c From the GDC \( HPC(t) = 2.67(1.59)^t \)

d Using the calculator, \( R^2 = 0.998 \)

e

f From the table, HPC > 50 for the first time on day 7. Therefore, he should apply the disinfectant every 7 days.

7 a \( N = 1000 \times 2^0 = 1000 \)

b \( 4000 = 1000 \times 2^{4k} \)
\( 2^{4k} = 4 = 2^2 \)
\( 4k = 2 \)
\( k = \frac{1}{2} \)

c \( N = 1000 \times 2^4 = 16000 \)

d \( 32000 = 1000 \times 2^{\frac{1}{2}t} \)
\( 2^{\frac{1}{2}t} = 32 = 2^5 \)
\( \frac{1}{2}t = 5 \)
\( t = 10 \text{ days} \)
Exercise 7J

1. a \( \log(100) = \log(10^2) = 2 \)
   
b \( \log(0.1) = \log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1 \)
   
c \( \ln(e) = 1 \)
   
d \( \ln\left(\frac{1}{e}\right) = \ln(e^{-1}) = -1 \)
   
e \( \ln(e^2) = 2 \)
   
f \( \log(\sqrt{10}) = \log\left(10^{1/2}\right) = \frac{1}{2} \)
   
g \( \ln\left(\frac{1}{\sqrt{e}}\right) = \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2} \)

2. a \( x = \log(10) = 1 \)
   
b \( x = \log(100) = 2 \)
   
c \( x = \log(38) \)
   
d \( x = \ln(e^2) = 2 \)
   
e \( x = \ln(3) \)
   
f \( x = \ln(0.3) \)
   
g \( x = \ln(1) = 0 \)

3. a \( \log(x^2) \)
   
b \( \log\left(x^{\frac{1}{2}}\right) = \log\left(\sqrt{x}\right) \)
   
c \( \log(x^3y^2) \)
   
d \( \log\left(\frac{x}{y^3}\right) \)
   
e \( \ln(x^2) = \ln\left(\frac{1}{x^2}\right) \)
   
f \( \log(10^2) + \log(x) = \log(100x) \)
   
g \( \ln\left(\frac{x}{y^2}\right) \)
4  a  The piecewise function must have the same value for $x = 1$ on both sides to be continuous. Thus, $4 \times (1)^3 - 3 = 2e^{a+(i)} \rightarrow a = \ln\left(\frac{1}{2}\right)$.

b  The piecewise function must have the same value for $x = 2$ on both sides to be continuous. Thus, $3 \times (2)^2 - 4 = 2\ln(a \times (2)) \rightarrow a = \frac{1}{2}e^4$.

Exercise 7K

1  a  i  $x = \log(3)$
ii  $x = \log(75)$
iii  $x = \ln(5)$

b  i  $e^x = \frac{15}{3} \Rightarrow x = \ln(5)$
ii  $10^x = 2 \times 2 \Rightarrow x = \log(4)$
iii  $e^x = 5 + 1 \Rightarrow x = \ln\left(\frac{8}{3}\right)$

2  a  $x = \log_a(c)$

b  $x = \ln\left(\frac{2}{b}\right)$

c  $x = \log\left(\frac{k}{2}\right)$

3  a  $10^{\log(x)} = 10^{\log(x^3)} = x^3$

b  $e^{\ln(x) - \ln(y)} = e^{\frac{\ln(x)}{y}} = \frac{x}{y}$

c  $10^{2\log(x) - \log(y)} = 10^{\log\left(\frac{x^2}{y}\right)} = \frac{x^2}{y}$

d  $e^{\ln\left(\frac{1}{x^2}\right)} = e^{\ln\left(x^{-2}\right)} = \frac{1}{x^2}$

4  a  $g(f(x)) = 2e^{\ln(x-2)} = 2(x + 2) = 2x + 4$

b  Vertical asymptote at $x = -2$

Intersection with $y$-axis at $(0, \ln(2))$

Intersection with $x$-axis at $\ln(x + 2) = 0 \Rightarrow x = -1$. Hence, it is at $(-1, 0)$
Hence, \( f^{-1}(x) = e^x - 2 \)

5 a \( y = 10^x - 3 \Rightarrow x = \log(y + 3) \), therefore \( f^{-1}(x) = \log(x + 3) \)

b Vertical asymptote: \( x = -3 \)

Intersection \( y \)-axis: \( (0, \log(3)) \)

Intersection \( x \)-axis: \( \log(x + 3) = 0 \Rightarrow x = -2 \), therefore \( (-2,0) \)

6 a Horizontal asymptote: \( y = -6 \)

Intersection \( y \)-axis: \( (0, 4) \)

Intersection \( x \)-axis: \( 2e^x - 6 = 0 \Rightarrow x = \ln(3) \), therefore \( (\ln(3),0) \)
b \[y = 2e^x - 6 \Rightarrow x = \ln\left(\frac{1}{2}(y + 6)\right)\] therefore \[f^{-1}(x) = \ln\left(\frac{1}{2}(x + 6)\right)\]

c The vertical asymptote is at \(x = -6\), so the domain is \(x > -6\). The range is all the real numbers, so \(f^{-1}(x) \in \mathbb{R}\).

Exercise 7L

1 a \[B = \log y = 3.2 \Rightarrow y = 10^{3.2} = 1585\]

\[G = \log y = 6.6 \Rightarrow y = 10^{6.6} = 3981072\]

\(G\) is 2512 times greater than \(B\).

b It goes from \(A \rightarrow y = 10^{1.7} = 50.12\) to \(J \rightarrow y = 10^{10.1} = 1.259 \times 10^{10}\).

c Using a log scale reduces the range and spreads the data points move evenly.

2 a

<table>
<thead>
<tr>
<th>Country</th>
<th>GNI</th>
<th>LE</th>
<th>log(GNI)</th>
<th>log(LE)</th>
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<tbody>
<tr>
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<td>4.62408</td>
<td>1.90687</td>
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</tr>
<tr>
<td>BGR</td>
<td>13980</td>
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<td>4.14551</td>
<td>1.8704</td>
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<tr>
<td>HRV</td>
<td>19330</td>
<td>77.1</td>
<td>4.28623</td>
<td>1.88705</td>
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<tr>
<td>CZE</td>
<td>24280</td>
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<td>4.38525</td>
<td>1.89098</td>
</tr>
<tr>
<td>CNK</td>
<td>42300</td>
<td>79.9</td>
<td>4.62634</td>
<td>1.90255</td>
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<tr>
<td>EST</td>
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<td>76.3</td>
<td>4.31869</td>
<td>1.88252</td>
</tr>
<tr>
<td>FIN</td>
<td>38500</td>
<td>80.3</td>
<td>4.58546</td>
<td>1.90472</td>
</tr>
<tr>
<td>FRA</td>
<td>35650</td>
<td>81.6</td>
<td>4.55206</td>
<td>1.91169</td>
</tr>
</tbody>
</table>
b  \[ y = ax^b \Rightarrow \log y = b \log x + \log a = 0.0795 \log x + 1.54 \]

c  In this linear model we get  \( \log a = 1.54 \Rightarrow a = 34.67 \),  \( b = 0.0795 \)

d  In the power model we get the same values, with any differences due to rounding during the process.

3  a  \( \ln t = \{0.875, 1.03, 1.22, 1.39, 1.63\} \)

b  

![Graph](image.png)
\[ t = ae^{bq} \Rightarrow \ln t = bq + \ln a \]

From the linear regression function on the GDC \(\ln a = 0.742\) and \(b = 0.0168\)

\[ \ln a = 0.744 \Rightarrow a = e^{0.744} = 2.10 \quad \text{giving the final model:} \quad t = 2.10e^{0.0168q} \]

\[ t(60) = 2.10e^{0.0168 \times 60} = 5.75 \quad \text{minutes} \]

4 a \( s(0) = c = \$1000 \)

\[ b \quad s - c = a \times n^b \Rightarrow \ln(s - c) = b \ln n + \ln a \]

\[ d \quad \ln(a) = 3.99, b = 0.75 \]

\[ e \quad \ln(s - 1000) = 3.99 + 0.75 \ln n \Rightarrow s(n) = 54.0 \times n^{0.75} + 1000 \]

\[ f \quad S(500) = \$6710 \]

\[ g \quad \text{The data only go up to 200 shirts; therefore, this is extrapolation far beyond the original data.} \]

\[ h \quad i \quad x = n, \ y = S - c \]

\[ ii \quad \text{This gives the same values for} \ a \ \text{and} \ b. \ \text{Any differences are due to rounding during the calculations.} \]

5 a 

\[ b \quad \text{We can get a system of equations such that:} \]

\[ 8.4 = a\sqrt{1} + b \Rightarrow a = 8.4 - b \]
12.9 = a\sqrt{4} + b \Rightarrow b = 12.9 - 2a

Substituting for $b$: $a = 8.4 - 12.9 + 2a \Rightarrow a = 4.5$ and $b = 12.9 - 2 \times 4.5 = 3.9$

**Exercise 7M**

1 a

![Graph](image)

b When $t = 0 \Rightarrow P(0) = \frac{100}{11} = 9.09\%$, which is the initial recorded percent of people with access to the Internet.

As $t \to \infty, P(t) \to \frac{100}{1} = 100\%$.

Hence range is $9.09 \leq P(t) < 100$

This means that we are tending asymptotically to a state where everyone will have the Internet, at least according to the model.

c We look for $P(t_{\text{half}}) = 50 = \frac{100}{1 + 10e^{-0.5t}} \Rightarrow t_{\text{half}} = -\frac{1}{0.5} \ln \left( \frac{100}{50} - 1 \right) = 4.6$ years.

d $P(20) = \frac{100}{1 + 10e^{0.5 \times 20}} = 99.95\%$

2 a In 1950 there are $P(0) = \frac{107000}{5} = 21400$ people.

b In 2015 there are $P(65) = \frac{107000}{1 + 4e^{-0.135 \times 65}} = 106934$ people.

c The limit population is when $t \to \infty, P(t) \to \frac{107000}{1 + 0} = 107000$ people.

3 a The behaviour at infinite should be 120 million so as $t \to \infty, P(t) \to L = 120000000 = 120 \times 10^6$.

Initially there are 10000 people infected, so $P(0) = \frac{120 \times 10^6}{1 + C} = 10000 \Rightarrow C = 11999$

After two weeks there are 20000 people infected, so $P(2) = \frac{120 \times 10^6}{1 + 11999 \times e^{-2\times k}}$

$k = -\frac{1}{2} \ln \left( \frac{1}{11999} \left( \frac{120 \times 10^6}{20000} - 1 \right) \right) = 0.347$. 

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Therefore, \( f(x) = \frac{120 \times 10^6}{1 + 11999 \times e^{-0.347x}} \).

b \( f(4) = 40000 \) infected to 2 significant figures.

c \( P(x_{90\%}) = 0.9 \times 120 \times 10^6 = \frac{120 \times 10^6}{1 + 11999 \times e^{-0.347 \times 90\%}} \Rightarrow x_{90\%} = -\frac{1}{0.347} \ln \left( \frac{1}{11999} \left( \frac{1}{0.9} - 1 \right) \right) = 33.4 \) weeks.

Chapter Review

1 a \( r = \frac{5}{2} = 2.5 \)

b \( u_b = 2 \times 2.5^7 = 1220.70 \)

c \( S_b = 2 \times \frac{2.5^7 - 1}{2.5 - 1} = 2033.17 \)

2 \( FV = 350(1 - 0.12)^5 = $184.71 \)

3 a This is a geometric series since the percentage increase is a constant. Thus,
\[
\frac{439230}{363000} = \frac{u_b}{u_1} = \frac{u_1 \times r^2}{u_1} = r^2 \Rightarrow r = \sqrt{\frac{439230}{363000}} = 1.1, \text{ so the percentage increase is } p = 10\%.
\]

b As it is the beginning of the third year two complete years have passed
\( 363000 = u_b \times 1.1^2 \Rightarrow u_b = €300000 \)

c At the beginning of the ninth year, eight complete years have passed
\( u_9 = 300000 \times 1.1^8 = €643077 \)

4 a First, we get the ratio \( r = \frac{6}{2} = 3 \). Now, we see which term is 118098 on the sequence:
\( u_n = 118098 = 2 \times 3^{n-1} \rightarrow N = \log_3 \left( \frac{118098}{2} \right) + 1 = 11 \). There are 11 terms in the sequence.

b \( S_{11} = 2 \times \frac{3^{11} - 1}{3 - 1} = 177146 \)

5 a At the beginning of the 2\textsuperscript{nd} year it would cost \( 9500(1 + 0.0116) = £9610.20 \)

b Starting with £9610.20 at the 2\textsuperscript{nd} year, the 3\textsuperscript{rd} year would be £9610.20(1 + 0.0114) = £9719.76

6 a \( P_6 = 3000 \left( 1 + \frac{0.0235}{12} \right)^{12 \times 6} = SGD3453.80 \)
or

\[
\begin{align*}
N &= 5000 = 3000 \left(1 + \frac{0.0235}{12}\right)^{12 \times N} \\
\Rightarrow N &= 21.8 \text{ years}
\end{align*}
\]

or

\[
\begin{align*}
N &= 21.75853805 \\
I &= 2.35 \\
PV &= -3000 \\
PMT &= 0 \\
FV &= 5000 \\
P/Y &= 1 \\
C/Y &= 12
\end{align*}
\]

Hence \( N = 21.8 \) years

\[
\begin{align*}
P &= 5179.27 = 4500 \left(1 + \frac{r}{4}\right)^{4 \times 6} \\
\Rightarrow r &= 4 \left(\frac{5179.27}{4500}\right)^{\frac{1}{24}} - 1 = 0.0235 = 2.35\%
\end{align*}
\]

or

\[
\begin{align*}
N &= 6 \\
I &= 2.349988186 \\
PV &= -4500 \\
PMT &= 0 \\
FV &= 5179.27 \\
P/Y &= 1 \\
C/Y &= 4
\end{align*}
\]

Interest is 2.35%
The payments are 115.36 euros per month.

10 a \( f (5) = 24000 \times 1.12^5 = 42296 \) rabbits

b \( f(t_o) = 48000 = 24000 \times 1.12^{t_o} \Rightarrow t_o = 6.12 \) years.

11 a \( T(0) = 21 + 74 = 95^\circ C \)

b \( T(10) = 21 + 74 \times 1.2^{-10} = 32.95^\circ C \)

c \( T(x_{40}) = 40 = 21 + 74 \times 1.2^{-x_{40}} \Rightarrow x_{40} = 7.46 \) minutes.

d The cup of soup will get to room temperature when \( x \to \infty \Rightarrow T(x) \to 21 + 0 = 21^\circ C \)

12 a \( y = 16 \)

b Cuts at \( x = 0 \), so \( f(0) = 4 + 16 = 20 \). Hence, the intersection point is \((0, 20)\).

13 a \( y(7) = 4 + e^7 = 1101 \) infected people.

b \( y(x_{25k}) = 25000 = 4 + e^{x_{25k}} \Rightarrow x_{25k} = \ln(25000 - 4) = 10.13 \) days. That is, at least 25000 people are infected after 11 days.

14 a \( h(4) = 0.25 + \log(2 \times 4 - 0.6) = 1.12 \) m

b \( h(t_2) = 2 = 0.25 + \log(2t_2 - 0.6) \Rightarrow t_2 = \frac{1}{2} \left( 10^{2.025 + 0.6} \right) \approx 28.42 \) weeks. That is, after 29 weeks the plant is more than 2 m tall.

15 a \( \sum_{r=1}^{n} 5 \times 2^r = 10 \times \frac{2^n - 1}{2 - 1} = 10 \times 2^n - 10 = 10 \left( 2^n - 1 \right) \)

b \( \sum_{r=1}^{2n} 5 \times 2^r = 10 \times \frac{2^{2n} - 1}{2 - 1} = 10 \left( 4^n - 1 \right) \)

16 Note than on this occasion the payments are made at the start of the time periods not the end.
Exam style questions

17 a 0.1kg

b \(0.05 = 0.1e^{-0.4t} \Rightarrow t = 1.73 \text{ years}\)

c \(0.01 = 0.1e^{-0.4t} \Rightarrow t = 5.76 \text{ years}\)

18 a \(\log xy = \log x + \log y = p + q\)

b \(\log \frac{x}{y} = \log x - \log y = p - q\)

c \(\log \sqrt{x} = \frac{1}{2} \log x = \frac{1}{2} p\)

d \(\log x^2y^5 = 2\log x + 5\log y = 2p + 5q\)

e \(\log x^y = y\log x = 10^p p\)

f \(\log 0.01x^3 = \log \left(\frac{x^3}{100}\right) = \log x^3 - \log 10^2 = 3p - 2\)

19 a \(\frac{a}{b+1} = 2, \frac{a}{b} = 3 \Rightarrow a = 6, b = 2\)

b \(h(10) = 2.53 \text{ m (3 sf)}\)

c \(t = -10\ln\left(\frac{a}{b+h}\right). \text{ For } h(t) = 2.8 \Rightarrow t = 19.5 \text{ years}\)

20 a i \(t = \frac{-6000}{\ln 2} \ln(0.5) = 6000 \text{ years}\) ii \(t = \frac{-6000}{\ln 2} \ln(0.25) = 12000 \text{ years}\)

b \(t = \frac{-6000}{\ln 2} \ln(0.01) = 40000 \text{ years}\)

c \(r = e^{\frac{\ln(2)}{1000}}\)

d \(r = e^{\frac{\ln(2)}{10000}} = 0.315(3 \text{ sf})\)

21 a \(ar^2 = 18, ar^5 = 486 \Rightarrow r^3 = 27 \Rightarrow r = 3, a = 2\)
\[ u_6 = 2 \times 3^2 = 4374 \]
\[ S_8 = 2 \left( \frac{3^8 - 1}{3 - 1} \right) = 6560 \]

\( 2 \times 3^{n-1} > 10^6 \), can be solved directly on the GDC or using logs, giving
\[ n > \log_3 \left( \frac{10^6}{2} \right) + 1 = 12.9 \] Hence, \( n = 13 \).

22a \( a \)
\[ u_n = ar^{n-1} \]
\[ S_n = a \left( \frac{r^n - 1}{r - 1} \right) \]
\[ v_n = \log(ar^{n-1}) = \log a + (n - 1)\log r \]
\( v_n \) is an arithmetic progression, with first term \( v_1 = \log a \) and common difference \( d = \log r \)
\[ T_n = \frac{n}{2} (2v_1 + (n-1)d) = \frac{n}{2} (2\log a + (n-1)\log r) \]
\[ \log S_n = \log \left( a \left( \frac{r^n - 1}{r - 1} \right) \right) = \log a + \log(r^n - 1) - \log(r - 1) \]
\[ = \frac{n}{2} (2\log a + (n-1)\log r) \] so answer is no.

23a \( a \)
\[ \frac{a}{1-r} = 3a \Rightarrow 1-r = \frac{1}{3} \Rightarrow r = \frac{2}{3} \]
\[ b \]
\[ \frac{a}{1-r} = \frac{2}{3} a \Rightarrow 1-r = \frac{3}{2} \Rightarrow r = -\frac{1}{2} \]
\[ c \]
\[ \frac{a}{1-r} = \frac{a}{3} \Rightarrow 1-r = 3 \Rightarrow r = -2 \] but sum to infinity does not exist in this case as the condition for sum to infinity to exist is \( |r| < 1 \). So the answer is no.

24a Require \( 1.03x = 100 \Rightarrow x = £97.09 \)
\[ b \] Require \( (1.03)^{10}x = 100 \Rightarrow x = £74.41 \)
\[ c \]
\[ N=10 \]
\[ I/X=3 \]
\[ PV=0 \]
\[ *PMT=-84.68981224 \]
\[ FV=1000 \]
\[ P/Y=1 \]
\[ C/Y=1 \]
\[ PMT:END \ BEG1N \]

Her payments will be \( M = £84.69 \)

25a \( 5000(1.04)^5 = 6083.26 \) euros
\[5000 \left(1 + \frac{0.038}{12}\right)^{60} = 6044.43 \text{ euros}\]

She should choose Scheme 1.

Part a can also be done using the Finance app on the GDC.

b \(6083.26 - 5000 = 1083.26 \text{ euros}\)

c For scheme 1,
\[r_1 = (1.04)^n\]
\[\ln r_1 = n \ln 1.04 = 0.039221n\]

For scheme 2,
\[r_2 = \left(1 + \frac{0.038}{12}\right)^{12n}\]
\[\ln r_2 = 12n \ln 1.0032 = 0.037940n\]

Since \(\ln r_1 > \ln r_2\), then scheme 1 is always the best.
Skills Check

1  a,b

Exercise 8A

1  a \[ \frac{30^\circ \pi}{180^\circ} = \frac{\pi}{6} \]
   b \[ \frac{165^\circ \pi}{180^\circ} = \frac{11\pi}{12} \]
   c \[ \frac{270^\circ \pi}{180^\circ} = \frac{3\pi}{2} \]
   d \[ \frac{300^\circ \pi}{180^\circ} = \frac{5\pi}{3} \]
   e \[ \frac{210^\circ \pi}{180^\circ} = \frac{7\pi}{6} \]

2  a \[ \frac{\pi}{3} \times \frac{180^\circ}{\pi} = 60^\circ \]
   b \[ \frac{4\pi}{3} \times \frac{180^\circ}{\pi} = 240^\circ \]
   c \[ \frac{3\pi}{5} \times \frac{180^\circ}{\pi} = 108^\circ \]
   d \[ 3\pi \times \frac{180^\circ}{\pi} = 540^\circ \]
   e \[ 1 \times \frac{180^\circ}{\pi} = 57.3^\circ \]

3  \( r = 7 \text{ m} \) and \( \theta = \frac{2\pi}{3} \):

   a Using the arc length formula \( L = r\theta \), we get that \( L = 7 \times \frac{2\pi}{3} = 14.7 \text{ m} \)

   b Using the arc area formula \( A = \frac{1}{2} r^2 \theta \), we get that \( A = \frac{1}{2} r^2 \times \frac{2\pi}{3} = 51.3 \text{ m}^2 \)

   c Area \( \frac{1}{2} r^2 \sin \theta = \frac{1}{2} r^2 \sin \frac{2\pi}{3} = 21.2 \text{ m}^2 \).

   d Area of segment = area of sector – area of triangle \( 51.3 - 21.2 = 30.1 \text{ m}^2 \).

4  a We sum the arc length of the curved part and the two radii \( r = 3 \text{ cm} \) to get the total perimeter: \( P = 3 + 3 + 3 \times 0.7 = 8.1 \text{ cm} \).

   \[ A = \frac{1}{2} r^2 \theta = \frac{1}{2} 3^2 \times 0.7 = 3.15 \text{ cm}^2 \].
b For the perimeter, the arc AB is given by $3a$, and the line segment AB is the base of the isosceles triangle AOB, which is $2 \times 7 \sin \left(\frac{\pi}{3}\right) = 12.1 \text{ m}$. Hence, the total perimeter is $P = 14.7 + 12.1 = 26.8 \text{ m}$. The area is the same as in $3d$, so it is $30.1 \text{ m}^2$.

c Arc length of AB: $r\theta = 6 \times 2 = 12 \text{ cm}$

Perimeter = $12 + 6 + 6 = 24 \text{ cm}$.

Area of sector = $\frac{1}{2} r^2 \theta = 36 \text{ cm}^2$

Area of triangle = $\frac{1}{2} \times 6^2 \sin(2) = 16.4 \text{ cm}^2$

Area of segment = $36 - 16.4 = 19.6 \text{ cm}^2$.

5 a As the opposite sides of the rectangle are parallel:

$\alpha = \sin^{-1} \left(\frac{2.5}{3}\right) = 56.4^\circ$ or $0.985 \text{ radians}$

b The area of grass is the area of the sector of the circle plus the area of the triangle:

Length of the base of the triangle is $\sqrt{3^2 - 2.5^2} = 1.66$

Area = $\frac{1}{2} \times 1.66 \times 2.5 + \frac{1}{2} \times 3^2 \times 0.985 = 6.51 \text{ m}^2$.

6

[Diagram of overlapping circles]

Let $\hat{CAD} = \alpha$ and $\hat{CBD} = \beta$.

a AB = 6 cm, AC = 3 cm, BC = 5 cm

For $\triangle ABC$,

$5^2 = 3^2 + 6^2 - 2 \times 3 \cos \left(\frac{\alpha}{2}\right)$

$\alpha = 2 \cos^{-1} \left(\frac{-20}{36}\right) = 1.96 \text{ rad}$

Similarly, $\beta = 1.04 \text{ rad}$

$P = r_\alpha \beta + r_\beta \alpha = 11.1 \text{ cm}$
b The area of the region is the area spanned by the arcs ACD and BCD, where
\[ A_{\text{segment}_{ACD}} = A_{\text{sector}_{ACD}} - A_{\text{triangle}_{ACD}} \]
and similarly for sector BCD.

Now, \[ A_{\text{sector}_{ACD}} = \frac{1}{2} r^2 \alpha = 8.82 \, \text{cm}^2, \]
\[ A_{\text{triangle}_{ACD}} = \frac{1}{2} 3^2 \sin 1.96 = 4.16 \, \text{cm}^2. \]

Thus, \[ A_{\text{segment}_{ACD}} = 8.82 - 4.16 = 4.66 \, \text{cm}^2. \]

Similarly, \[ A_{\text{segment}_{BCD}} = 13.0 - 10.78 = 2.22 \, \text{cm}^2. \]

Hence, the total area is \[ A_{\text{shaded}} = 4.66 + 2.22 = 6.88 \, \text{cm}^2. \]

Note that \( \alpha > 180^\circ \). \( AB = AC = AD = 2 \, \text{cm}. \) \( BC = BD = 3 \, \text{cm}. \)

The shaded area is the sum of the sector CBD of radius 3 cm and angle \( \beta \) plus the two segments for chords CB and BD.

First, we get the angles \[ \frac{\alpha}{2} = \angle C\hat{A}B = 2 \times \sin^{-1} \left( \frac{BE}{AB} \right) = 2 \times \sin^{-1} \left( \frac{3/2}{2} \right) = 1.70 \, \text{rad} \] and
\[ \frac{\beta}{2} = \angle C\hat{A}B = \cos^{-1} \left( \frac{BE}{AB} \right) = \cos^{-1} \left( \frac{3/2}{2} \right) = 0.72 \, \text{rad}. \]

Thus, the area of sector CBD is simply \[ \frac{1}{2} \times 3^2 \times (2 \times 0.72) = 6.48 \, \text{cm}^2. \]

Sector CAB is congruent to sector DAB.
\[ A_{\text{sector}_{CAB}} = \frac{1}{2} \times 2^2 \times 1.70 = 3.40 \] and \[ A_{\text{triangle}_{CAB}} = \frac{1}{2} \times 3^2 \sin(1.70) = 1.98 \, \text{cm}^2. \]

Thus, \[ A_{\text{segment}_{CAB}} = 3.40 - 1.98 = 1.42 \, \text{cm}^2. \]

Hence the total shaded area is \[ A_{\text{shaded}} = 6.48 + 2 \times 1.42 = 9.32 \, \text{cm}^2. \]

Exercise 8B

1 a 0 b 1 c 0 d 0
e −1 f −1 g −1 h 0

2 a There are two solutions per cycle. These sum half a cycle in the case of the sine and a whole cycle in the case of the cosine. As \( 0 \leq x \leq 720 \), there will be four solutions on each exercise.

i \[ x = \{10^\circ, 170^\circ, 370^\circ, 530^\circ\} \]

ii The first solution is \[ x = \sin^{-1}(0.3) = 17.5^\circ. \] Hence, \[ x = \{17.5^\circ, 162.5^\circ, 377.5^\circ, 522.5^\circ\} \]
iii \( x = \{160^\circ, 200^\circ, 520^\circ, 560^\circ\} \)

b  \( 0 \leq x \leq 3\pi \), so there may be 3 or 4 solutions on each exercise.

i \[ x = \left\{ \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{12\pi}{5}, \frac{13\pi}{5} \right\} \]

ii \( x = \{1.67, 4.61, 7.95\} \)

iii \[ x = \left\{ \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \right\} \]

3 a By inspecting the graph, one can find the points of intersection:

i \((x, y) = (0.51, 0.49)\)

ii \((x, y) = \{ (0.95, 0.81), (2.65, 0.47), (6.07, -0.21)\} \)

b i The graph shows that there is only one solution, because range of \( y = \sin x \) is \(-1 \leq y \leq 1\) so there can be no more intersections.

ii The graph shows that there are three solutions, because range of \( y = \sin x \) is \(-1 \leq y \leq 1\)
The two points with angles $\theta$ and $180 - \theta$ are shown as the points B and A on the diagram. Point A has coordinates (-x, y) and point B coordinates (x, y). As they have the same y coordinate we can determine that $y = \sin \theta = \sin(180 - \theta)$.

b i $\sin(\theta) = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \times \frac{180}{\pi} = 30^\circ$

ii $\sin(30^\circ) = \sin(180^\circ - 30^\circ) = \sin(150^\circ) = \frac{1}{2}$

c i Using the sine rule: $\frac{\sin(20^\circ)}{3} = \frac{\sin(B\hat{C}A)}{5} \Rightarrow \sin(B\hat{C}A) = \frac{5\sin(20^\circ)}{3}$.

ii $\sin(B\hat{C}A) = 0.57 \Rightarrow B\hat{C}A = 34.8^\circ$.

iii $\sin(34.8^\circ) = \sin(180^\circ - 34.8^\circ) = \sin(145.2^\circ) \Rightarrow B\hat{C}A = 145^\circ$

iv

5 There are two possible angles $\hat{R}$, corresponding to the possible lengths $PR$. We apply the sine rule $\frac{\sin(25^\circ)}{5} = \frac{\sin(\hat{R}_1)}{7} \Rightarrow \hat{R}_1 = \sin^{-1}\left(\frac{7\sin(25^\circ)}{5}\right) = 36.28^\circ$. The other angle is $\hat{R}_2 = 180^\circ - \hat{R}_1 = 143.72^\circ$. Here $\hat{Q}_1 = 180^\circ - 25^\circ - \hat{R}_1$, as the internal angles of a triangle sum to $180^\circ$. So, $\hat{Q}_1 = 118.72^\circ$ and $\hat{Q}_2 = 11.28^\circ$. We use the sine rule again to get the values of $PR$.

$$\frac{\sin(\hat{R}_1)}{7} = \frac{\sin(\hat{Q}_1)}{PR_1}$$

$$PR_1 = 7 \times \frac{\sin(\hat{Q}_1)}{\sin(\hat{R}_1)}$$

Hence $PR_1 = 10.4$ cm and $PR_2 = 2.31$ cm.

6 a We notice that $h = \sqrt{2^2 - 1^2} = \sqrt{3}$

i $\frac{\sqrt{3}}{2}$  

ii $\frac{1}{2}$  

iii $\frac{1}{2}$  

iv $\frac{\sqrt{3}}{2}$
Worked solutions

b i \( \cos(x) = \cos(-x) \) so \( \cos(-60^\circ) = \frac{1}{2} \)

ii \( \sin\left(\frac{\pi}{6} \times \frac{180}{\pi}\right) = \sin(30^\circ) = \frac{1}{2} \)

iii \( \cos\left(\frac{\pi}{6} \times \frac{180}{\pi}\right) = \cos(-30^\circ) = \frac{\sqrt{3}}{2} \)

iv \( \sin(-x) = -\sin(x) \) so \( \sin\left(-\frac{\pi}{3} \times \frac{180}{\pi}\right) = -\sin(60^\circ) = -\frac{\sqrt{3}}{2} \)

7 a i As \( \cos(x) = \frac{1}{3} \), then \( \sin^2(x) + \left(\frac{1}{3}\right)^2 = 1 \rightarrow \sin(x) = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3} \)

ii As \( \tan(x) = \frac{\sin(x)}{\cos(x)} \) \( \Rightarrow \tan(x) = \frac{2\sqrt{2}}{3} = 2\sqrt{2} \)

b i \( \cos(-x) = \frac{1}{3} \)

ii \( \sin(-x) = -\frac{2\sqrt{2}}{3} \)

8 a We know \( \tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\sin(x)}{2 \sin(x)} = \frac{1}{2} \)

b i We know \( \sin^2(x) + \cos^2(x) = 1 \rightarrow \sin^2(x) + 4 \sin^2(x) = 1 \). Hence \( \sin(x) = \frac{1}{\sqrt{5}} \)

ii As \( \cos(x) = 2 \sin(x) \), then \( \cos(x) = \frac{2}{\sqrt{5}} \)

c Because in this domain \( \sin(x) \) would be negative and \( \cos(x) \) positive, so the equality \( 2 \sin(x) = \cos(x) \) would no longer hold.

Exercise 8C

1 a \( y = 2 \sin(3(x - 4)) - 5 \)

i \( a = 2 \) ii \( y = -5 \) iii \( P = \frac{2\pi}{3} \) iv \( c = 4 \) to the right

b \( y = -3 \cos\left(\pi(x + 1)\right) + 3 \)

i \( a = 3 \) ii \( y = 3 \) iii \( P = \frac{2\pi}{\pi} = 2 \) iv \( c = 1 \) to the left

c \( y = 4 \sin(3x - 6) = 4 \sin(3(x - 2)) + 0 \)

i \( a = 4 \) ii \( y = 0 \) iii \( P = \frac{2\pi}{3} \) iv \( c = 2 \) to the right

d \( y = \cos(2x + 5) - 1 = \cos\left(2 \left(x + \frac{5}{2}\right)\right) - 1 \)

i \( a = 1 \) ii \( y = -1 \) iii \( P = \frac{2\pi}{2} = \pi \) iv \( c = \frac{5}{2} \) to the left
2 a \[ a = \frac{1}{2}(3 - (-1)) = 2, \quad d = \frac{1}{2}(3 + (-1)) = 1, \quad P = 4 - 0 = 4 \Rightarrow b = \frac{2\pi}{P} = \frac{\pi}{2}. \]
Hence \[ y = 2\sin\left(\frac{\pi}{2} x\right) + 1 \]

b \[ a = \frac{1}{2}(1.5 - (-3.5)) = 2.5, \quad d = \frac{1}{2}(1.5 + (-3.5)) = -1, \quad P = 6 - 0 = 6 \Rightarrow b = \frac{2\pi}{P} = \frac{\pi}{3}. \]
Hence \[ y = 2.5\sin\left(\frac{\pi}{3} x\right) - 1 \]

3 a i \[ a = \frac{1}{2}(3 - (-5)) = 4, \quad d = \frac{1}{2}(3 + (-5)) = -1, \quad P = 6.5 - 1.5 = 4 \Rightarrow b = \frac{2\pi}{P} = \frac{\pi}{2}, \]
\[ c = -0.5 + 1.5 = \frac{1}{2}. \]
Hence, \[ y = 4\sin\left(\frac{\pi}{2}(x - \frac{1}{2})\right) - 1 \]
i i \[ a = \frac{1}{2}(5 - (-1)) = 3, \quad d = \frac{1}{2}(5 + (-1)) = 2, \quad P = 2(6.5 - 3.5) = 6 \Rightarrow b = \frac{2\pi}{P} = \frac{\pi}{3}, \]
\[ c = 0.5 + 3.5 = 2. \]
Hence, \[ y = 3\sin\left(\frac{\pi}{3}(x - 2)\right) + 2 \]

b i As \( \sin(x) = \cos\left(x - \frac{\pi}{2}\right) \), then \[ y_i = 4\cos\left(\frac{\pi}{2}(x - \frac{1}{2}) - \frac{\pi}{2}\right) - 1 = 4\cos\left(\frac{\pi}{2}(x - \frac{3}{2})\right) - 1 \]
i i \[ y_i = 3\cos\left(\frac{\pi}{3}(x - 2) - \frac{\pi}{2}\right) + 2 = 3\cos\left(\frac{\pi}{3}(x - \frac{7}{2})\right) + 2. \]

c The translation will be one quarter of the period to the left. Hence \( k = \frac{1}{4} \)

4 a Dates are not real numbers, whereas the day numbers are.

b Each time is represented as an hours part \((h)\) and a minutes part \((m)\). Therefore, “hours after midnight” is \( h + \frac{m}{60} \).

c Note: some calculators may give slightly different answers
\[ f(t) = 1.5\sin(0.017t + 1.67) + 6.4 \]

d Notice that 02-Feb-2019 is day 398 in our scale. Hence, \( f(398) = 7.65 = 7:39 \)

e The times of sunrise for each date in subsequent years are nearly the same, so whilst this is extrapolation, it will be reliable for future dates.

5 a We know \( b = \frac{2\pi}{P} = 2\pi f \), hence, \( b = 1000\pi = 3142 \)

b 1ms = 0.001s, hence, \( S_2(t) = 4\sin(1000\pi(t - 0.001)) \)
c. Looks like a sinewave which is on counterphase with roughly half the amplitude.

b. By inspection we find that max \((0.00825, 4.98)\), min \((0.02396, -4.98)\)

c. i. \(a = 4.98\)

ii. Distance between maximum and minimum values is one half of the period so period = 
\[2 \left(0.02396 - 0.00825\right) = 0.0314\]

d. \(b = \frac{2\pi}{0.03142} = 200\) For the delay, one sees the first crossing with the principal axis which, from the GDC is \(c \approx 0.00040\). Hence, 
\[S_T(t) = 4.98\sin(200(t - 0.0004)) + 0\]

7. a. \(h_{\text{max}} = 4.1, \ h_{\text{min}} = 0.8\) Thus, \(a = \frac{1}{2}(4.1 - 0.8) = 1.65, \ d = \frac{1}{2}(4.1 + 0.8) = 2.45\)

b. We take the mean between the time difference on the highest and lowest tides to get the period: \(P = \frac{1}{2}\left((15.15 - 3.033) + (21.32 - 8.9)\right) = 12.27\). Hence, \(b = \frac{2\pi}{P} = 0.512\)

c. We know that \(\sin(x)\) has a first maximum at \(x = \frac{\pi}{2}\). The first maximum tide height happens at \(t_{\text{max}} = 3.033\), so we want that \(0.512(t_{\text{max}} - c) = \frac{\pi}{2} \Rightarrow c = 3.033 - \frac{\pi}{2 \times 0.512} = -0.035\). Hence, the model is \(\text{height}(t) = 1.65\sin(0.512(t + 0.035)) + 2.45\)
d The regression produces the equation \( f(t) = 1.54 \sin(0.518(t - 0.03)) + 2.5 \) which is similar to the model we got by inspection of the table.

e Using the rounded values from the answer to part c we obtain the following table (though answers will differ if more figures are used).

<table>
<thead>
<tr>
<th>Time, t</th>
<th>Height, h(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>07:52</td>
<td>1.15 m</td>
</tr>
<tr>
<td>10:24</td>
<td>1.12 m</td>
</tr>
<tr>
<td>20:08</td>
<td>1.16 m</td>
</tr>
<tr>
<td>22.49</td>
<td>1.19 m</td>
</tr>
</tbody>
</table>

The lowest depth of water that is rowable is about 1.12 m.

Exercise 8D

1 a \( x = \frac{-4 \pm \sqrt{16 - 8}}{4} = 1 \pm \frac{1}{\sqrt{2}} = 1.71 \) or 0.29

b We factorise as \((x - 5)(x + 1) = 0\), thus \( x = 5 \) or \( x = -1 \)

c \( x = \frac{8 \pm \sqrt{64 - 80}}{8} = 1 \pm \frac{\sqrt{-16}}{8} = 1 \pm \frac{1}{2} \)

d \( x = \pm \sqrt{10} = \pm \sqrt{10} \)

2 a \( 2a - 3b = 4 + 2i - 9 + 6i = -5 + 8i \)

b \( ab = (2 + i)(3 - 2i) = 6 + 2i - 6i - 2 = 4 - 4i = 8 - i \)

c \( \frac{a}{b} = \frac{ab^*}{|b|^2} = \frac{1}{3^2 + 2^2} (2 + i)(3 + 2i) = \frac{1}{13} (6 - 2i + 3i + 4i) = \frac{1}{13} (4 + 7i) \)

d \( b^2 = (3 - 2i)(3 - 2i) = 9 - 4 - 12i = 5 - 12i \)

e \( c^3 = c^2c = (1 - 1 - 2i)(1 - i) = -2i(1 - i) = -2 - 2i \)

f \( \frac{a^4}{b} = \frac{a}{b} a^3 \). From 2 c, \( \frac{a}{b} = \frac{1}{13} (4 + 7i) \). \( a^3 = a^2a = (3 + 4i)(2 + i) = 2 + 11i \).

Hence \( \frac{a^4}{b} = \frac{1}{13} (4 + 7i)(2 + 11i) = \frac{1}{13} (-69 + 58i) \)

3 Using the quadratic formula \( x = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i \), hence,

\( x^2 - 4x + 5 = (x - (2 + i))(x - (2 - i)) \)

4 The other root is \( 3 + i \) hence the equation can be written as \((x - (3 + i))(x - (3 - i)) = 0 \)

This expands to give \( x^2 - (3 + i)x - (3 - i)x + (3 - i)(3 + i) = x^2 - 6x + 10 \)
Exercise 8E

1

\[ a \quad 8\text{cis}\left(\frac{\pi}{2}\right) \]

\[ b \quad 7\text{cis}\pi \]

\[ c \quad 12\text{cis}0 \]

\[ d \quad 5\text{cis}\left(-\frac{\pi}{2}\right) \]

2

\[ a \quad r = \sqrt{13} \approx 3.61, \quad \theta = \tan^{-1}\left(\frac{3}{2}\right) = 0.983 \text{ but fourth quadrant so } -0.983 \]

\[ b \quad r = \sqrt{29} \approx 5.39, \quad \theta = \tan^{-1}\left(\frac{5}{2}\right) = 1.19 \text{ rad} \]

\[ c \quad r = \sqrt{10} \approx 3.16, \quad \theta = \tan^{-1}\left(\frac{1}{3}\right) = 0.322 \text{ rad but 4th quadrant so } -0.322 \]

\[ d \quad r = \sqrt{20} \approx 4.47, \quad \tan^{-1}\left(\frac{1}{2}\right) = 0.464 \text{ rad, but 3rd quadrant so } \theta = 0.464 - \pi = -2.68 \text{ rad} \]

\[ e \quad r = \sqrt{29} \approx 5.39, \quad \tan^{-1}\left(\frac{2}{5}\right) = 0.381 \text{ rad, but 2nd quadrant so } \theta = \pi - 0.381 = 2.76 \text{ rad} \]

\[ f \quad r = \sqrt{10} \approx 3.16, \quad \tan^{-1}\left(3\right) = 1.25 \text{ rad but 4th quadrant so } -1.25 \]

3 We use that \( \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \quad \tan\left(\frac{\pi}{4}\right) = 1, \quad \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \).

\[ a \quad r = \sqrt{4^2 + 4^2} = 4\sqrt{2}, \quad \theta = \tan^{-1}(1) = \frac{\pi}{4} \rightarrow z = 4\sqrt{2} \text{cis}\left(\frac{\pi}{4}\right) \]

\[ b \quad r = \sqrt{2^2 + 3^2} = \sqrt{13} = 4, \quad \theta = \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{3} \rightarrow z = 4\text{cis}\left(\frac{\pi}{3}\right) \]

\[ c \quad r = \sqrt{3^2 + 3^2} = \sqrt{18} = 6, \quad \theta = \tan^{-1}\left(-\frac{1}{3}\right) = -\frac{\pi}{6} \rightarrow z = 6\text{cis}\left(-\frac{\pi}{6}\right) \]

\[ d \quad r = \sqrt{3 + 1} = 2, \quad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}, \text{ but 3rd quadrant so } \theta = \frac{\pi}{6} - \pi = -\frac{5\pi}{6} \quad \text{rad } \Rightarrow z = 2\text{cis}\left(-\frac{5\pi}{6}\right) \]
\[ e \quad r = \sqrt{5^2 + 5^2} = 5\sqrt{2}, \ \tan^{-1}(-1) = -\frac{\pi}{4}, \ \text{but 2nd quadrant so} \]
\[ \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \ \text{rad} \Rightarrow z = 5\sqrt{2} \ cis\left(\frac{3\pi}{4}\right) \]

\[ f \quad r = \sqrt{7^2 + 7^2 \times 3 = \sqrt{7^2 \times 4} = 14, \ \theta = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3} \Rightarrow z = 14 \ cis\left(\frac{\pi}{3}\right) \text{ as it is in the 4th quadrant} \]

4  \[ a \quad 3cis(60^\circ) = 1.5 \pm 2.60i \]
\[ b \quad 4cis(120^\circ) = -2 \pm 3.46i \]
\[ c \quad 2cis(-150^\circ) = -1.73 - i \]
\[ d \quad 5cis(0.4) = 4.61 \pm 1.95i \]
\[ e \quad 2.4cis(1.9) = -0.78 \pm 2.27i \]
\[ f \quad 3.8cis(-0.6) = 3.14 \pm 2.15i \]

**Exercise 8F**

1  We will use  \( a = 1 + \sqrt{3}i \), \( b = -1 + i \), \( c = \sqrt{3} - i \).

\[ a \quad |a| = \sqrt{1 + 3} = 2, \ \theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3} \Rightarrow a = 2 \ cis\left(\frac{\pi}{3}\right) \]
\[ |b| = \sqrt{1 + 1} = \sqrt{2}, \ \tan^{-1}(1) = \frac{\pi}{4}, \ \text{but 2nd quadrant so} \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \Rightarrow b = \sqrt{2} \ cis\left(\frac{3\pi}{4}\right) \]
\[ |c| = \sqrt{3 + 1} = 2, \ \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \Rightarrow c = 2 \ cis\left(-\frac{\pi}{6}\right) \text{ as it is in the 4th quadrant} \]

\[ b \quad ac = 4e^{i(\pi/3 - \pi/6)} = 4e^{i\pi/6}, \ \text{so} \ x = 4 \times \sqrt{3} / 2 = 2\sqrt{3} \ \text{and} \ y = 4 \times 1 / 2 = 2 \]
\[ ac = 2\sqrt{3} + 2i \]
\[ ii \quad \frac{a}{c} = \frac{2e^{i\pi/3}}{2e^{-i\pi/6}} = e^{i(\pi/3 + \pi/6)} = e^{i\pi/2} = i \]
\[ iii \quad b^4 = \sqrt{2}^4 \ cis\left(4 \times \frac{3\pi}{4}\right) = 4 \ cis(3\pi) = 4 \ cis(\pi) = -4 \]
\[ iv \quad a^3 = 2^3 \ cis\left(3 \times \frac{\pi}{3}\right) = 8 \ cis(\pi) = -8, \ b^2 = \sqrt{2}^2 \ cis\left(2 \times \frac{3\pi}{4}\right) = 2 \ cis\left(\frac{3\pi}{2}\right) = -2i. \ \text{Thus,} \]
\[ \frac{a^3}{b^2} = \frac{-8}{-2i} = 4i \]
\[ v \quad \frac{a^2 b}{c^2} = \left(\frac{a}{c}\right)^2 \ b = i^2 \ b = -b = 1 - i \]

2  We first find the argument of \( z \)  \( \arg(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \) or 30\(^\circ\)
Thus \( z^n = 2^n \left( \cos \left( \frac{n \pi}{6} \right) + i \sin \left( \frac{n \pi}{6} \right) \right) \). This will have imaginary part 0 when

\[
\sin \left( \frac{n \pi}{6} \right) = 0 \Rightarrow n \frac{\pi}{6} = k \pi, \text{ where } k \text{ is an integer. Hence } n = 6k \text{ where } k \in \mathbb{Z}, \text{ i.e. }\]

\( n = \{-6, 0, 6, 12, 18, 24\ldots\} \)

3  a  \( r = \sqrt{2^2 + 2^2} = 2, \ \theta = \arctan(1) = \frac{\pi}{4} \rightarrow z = 2e^{i\theta} \)

b  3 + 3i can be plotted as the coordinate (3,3) in the Argand diagram.

c Points plotted on the Argand diagram in part b

d  u is a translation of \( \left( \frac{\sqrt{2}}{\sqrt{2}} \right) \).

v is a rotation of \( \frac{\pi}{4} \) anticlockwise and an enlargement by a factor of 2.

s is a rotation of \( \frac{\pi}{4} \) clockwise and an enlargement by a factor of 0.5.

4  If \( z = x + iy \), then \( z^* = x - iy \). Thus, \( |z^*| = |z| = r \) and

\[
\arg(z^*) = \tan^{-1} \left( -\frac{y}{x} \right) = -\tan^{-1} \left( \frac{y}{x} \right) = -\arg(z) = -\theta. \text{ Hence } z^* = re^{-i\theta}.
\]

\( zz^* = r^2 e^{i\theta - i\theta} = r^2 \) and therefore \( \text{Im}(zz^*) = 0 \)

5  \( e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 \rightarrow e^{i\pi} + 1 = 0 \).

6  a  Recall that \( \tan \left( \frac{\pi}{4} \right) = 1 \rightarrow \sin \left( \frac{\pi}{4} \right) = \cos \left( \frac{\pi}{4} \right) \). So, the line described by \( \arg(z) = \frac{\pi}{4} \) is the line

where \( \text{Re}(z) = \text{Im}(z) \), which is the line with slope 1 in the Argand diagram.
b We know \( z(t) = 0.5te^{zt} \)

i Crossing the imaginary axis happens when \( \text{Re}(z) = 0 \rightarrow \cos(\theta) = 0 \rightarrow \theta = \frac{\pi}{2} \). Hence
\[
\frac{\pi}{2} t = \frac{\pi}{2} \rightarrow t = 1.
\]

ii Crossing the real axis happens when \( \text{Im}(z) = 0 \rightarrow \sin(\theta) = 0 \rightarrow \theta = \pi \).

Hence \( \frac{\pi}{2} t = \pi \rightarrow t = 2 \).

c Imaginary crossing: \( z_1 = 0 + 0.5i \), real crossing: \( z_2 = 1 + 0i \).

d We know that all \( z_1, z_2, z_3 \) have argument \( \theta \). Thus, the intersection points will happen when
\[
0.5te^{zt} = r_1e^{\theta}, r_2e^{i(\theta+2\pi)} \text{ and } r_3e^{i(\theta+4\pi)}
\]

The argument for A will be \( \theta \) so \( \frac{\pi}{2} t = \theta \Rightarrow t = \frac{2\theta}{\pi} \) and the value of A is
\[
0.5 \times \frac{2\theta}{\pi} e^{i\theta} = \frac{\theta}{\pi} e^{i\theta}
\]

The curve will next intersect the line when \( \frac{\pi}{2} t = 2\pi + \theta \Rightarrow t = \frac{2(2\pi + \theta)}{\pi} = 4 + \frac{2\theta}{\pi} \)

Hence the value of B is \( \left(2 + \frac{\theta}{\pi}\right)e^{i(2\pi + \theta)} = \left(2 + \frac{\theta}{\pi}\right)e^{i\theta} \)

The curve will next intersect the line when \( \frac{\pi}{2} t = 4\pi + \theta \Rightarrow t = \frac{2(4\pi + \theta)}{\pi} = 8 + \frac{2\theta}{\pi} \)

Hence the value of B is \( \left(4 + \frac{\theta}{\pi}\right)e^{i(4\pi + \theta)} = \left(4 + \frac{\theta}{\pi}\right)e^{i\theta} \)

Since \( |z_1| - |z_2| = \left(2 + \frac{\theta}{\pi}\right) - \left(\frac{\theta}{\pi}\right) = 2, \quad |z_1| - |z_3| = \left(4 + \frac{\theta}{\pi}\right) - \left(2 + \frac{\theta}{\pi}\right) = 2 \).

Exercise 8G

1 Let \( V_1 = 110\sin(ax) \) and \( V_2 = 110\sin(ax + 60^\circ) \). We know the maximum output of a sine wave is just its modulus, so
\[
\max(V_i) = |V_i + V_2| = |110(\text{cis}(0) + \text{cis}(60^\circ))| = 110\left|\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right| = 110\sqrt{\frac{3}{4}} = 110\sqrt{3} \text{ V}.
\]
2  a Let \( V_1 = 110 \sin(at + 0) = 110 \text{Im}(\text{cis}(at + 0)) = 110 \text{Im}(\text{cis}(at)\text{cis}(0)) \). Similarly, we define
\[
V_2 = 110 \text{Im}\left(\text{cis}(at)\text{cis}\left(\frac{2\pi}{3}\right)\right) \quad \text{and} \quad V_3 = 110 \text{Im}\left(\text{cis}(at)\text{cis}\left(\frac{4\pi}{3}\right)\right),
\]
which are separated with a phase of \( \frac{2\pi}{3} \) each.

Thus,
\[
V_1 + V_2 + V_3 = 110 \text{Im}\left(\text{cis}(at)\left(\text{cis}(0) + \text{cis}\left(\frac{2\pi}{3}\right) + \text{cis}\left(\frac{4\pi}{3}\right)\right)\right) = 110 \text{Im}\left(\text{cis}(at)\left(1 - 0.5 + i\frac{\sqrt{3}}{2} - 0.5 - i\frac{\sqrt{3}}{2}\right)\right) = 110 \text{Im}(\text{cis}(at)(0 - 0)) = 0
\]

b Connecting in reverse is equivalent to reflecting in the x-axis as all positive becomes negative and vice versa. That is the same as shifting by 180°.

c In the shifted case
\[
-V_1 + V_2 + V_3 = 110 \text{Im}\left(\text{cis}(at)\left(-1 - 0.5 + i\frac{\sqrt{3}}{2} - 0.5 - i\frac{\sqrt{3}}{2}\right)\right) = 220 \text{Im}(\text{cis}(at)) = 220 \sin(at)
\]

3 Given \( V_1 = 6 \sin(at) \) and \( V_2 = 10 \sin(at + 40°) \), the combined maximum output is
\[
|V_1 + V_2| = |6 \text{cis}(0) + 10 \text{cis}(40°)| = |6 + 7.66 + i6.43| = 15.1 V.
\]

4 \( r = \left|3\text{cis}\left(\frac{\pi}{12}\right) + 4\text{cis}\left(\frac{3\pi}{4}\right)\right| = |2.90 + 0.78i - 2.83 + 2.83i| = |0.07 + 3.61i| = 3.61 \) and
\[
\alpha = \arg\left(3\text{cis}\left(\frac{\pi}{12}\right) + 4\text{cis}\left(\frac{3\pi}{4}\right)\right) = \arg(0.07 + 3.61i) = \tan^{-1}\left(\frac{3.61}{0.07}\right) = 1.55. \text{ Thus,}
\]
\[
f(x) + g(x) = 3.61 \sin(3x + 1.55)
\]

5  a The length of the day is simply \( h(t) = g(t) - f(t) \),
\[
h(t) = g(t) - f(t) = 2.19 \sin(0.0165t - 1.23) - 2.14 \sin(0.0165t + 1.81) + 18 + 5.97
\]
\[
= 2.19 \sin(0.0165t - 1.23) - 2.14 \sin(0.0165t + 1.81) + 12.03
\]
\[
h(t) = \text{Im}(2.19e^{0.0165t-1.23i} - 2.14e^{0.0165t+1.81i}) + 12.03
\]
\[
= \text{Im}(e^{0.0165t}(2.19e^{-1.23i} - 2.14e^{1.81i})) + 12.03
\]
\[
= \text{Im}(e^{0.0165t} \times 4.32e^{-1.28i}) + 12.03
\]
\[
= 4.32 \sin(0.0165t - 1.28) + 12.03
\]

b Longest day is \( 4.32 + 12.03 = 16.35 \) hours

Shortest day is \( -4.32 + 12.03 = 7.71 \) hours

c From a graph on the GDC or by solving the equation \( \sin(0.0165t - 1.28) = 1 \) the longest day will be when \( t = 172.8 \), which is 21 June, while the shortest one will be when
\[
\sin(0.0165t + \alpha) = -1 \Rightarrow 0.0165t + \alpha = \frac{3\pi}{2}, \text{ hence } t = 363.2, \text{ which is 29 December.}
Chapter review

1. $A_{\text{shaded}} = A_{\text{circle}} - A_{\text{not shaded}}$, where $A_{\text{not shaded}} = \frac{1}{2}A_{\text{circle}} + A_{\text{segment}}$, where

$A_{\text{segment}} = A_{\text{sector}} - A_{\text{triangle}}$.

Let’s get all of this in terms of $R$:

$A_{\text{triangle}} = \frac{1}{2} (b \times h) = \frac{1}{2} (AB + R) = \frac{1}{2} (2R \times R) = R^2$

$A_{\text{sector}} = \frac{1}{2} AP \times \pi = \frac{1}{2} \sin \frac{\pi}{2}$, where $AP^2 = OP^2 + OA^2 = 2R^2$, using Pythagoras theorem. Thus,

$A_{\text{sector}} = \frac{\pi}{2} R^2$

Hence, $A_{\text{segment}} = \frac{\pi}{2} R^2 - R^2 = \left(\frac{\pi}{2} - 1\right) R^2$

Now, $A_{\text{circle}} = \pi R^2$

Hence, $A_{\text{not shaded}} = \frac{\pi}{2} R^2 + \left(\frac{\pi}{2} - 1\right) R^2 = (\pi - 1) R^2$

Finally, $A_{\text{shaded}} = \pi R^2 - (\pi - 1) R^2 = R^2$.

2. a. Area $= \frac{1}{2} \times b \times c \times \sin A = \frac{1}{2} \times 1 \times 2 \times \sin \left(\frac{2\pi}{3}\right) = 0.866$

b. $BC^2 = 2^2 + 1^2 - 2 \times 2 \times 1 \times \cos \left(\frac{2\pi}{3}\right) = 7 \Rightarrow BC = 2.65$

3. The perimeter is the sum of the three sides, so $P = 5 + 5 + 5 \times 2 = 20$ cm. The area is just the arc sector $A = \frac{1}{5} \times 2 \times 2 = 25$ cm$^2$.

4. $\sqrt{3} + i = r e^{i\theta}$, where $r = \sqrt{3} + 1 = 2$ and $\theta = \arctan \left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. Thus, $2 \cos \left(\frac{\pi}{6}\right) = 2 \cos \left(\frac{10\pi}{6}\right)$, so $\arg \left(\sqrt{3} + i\right) = \frac{5\pi}{3}$ or $-\frac{\pi}{3}$

5. $|z_1| = \sqrt{1 + 1} = \sqrt{2}$, $\arg (z_1) = \tan^{-1}(1) = \frac{\pi}{4}$. Hence $z_1 = \sqrt{2} \cos \left(\frac{\pi}{4}\right)$

$|z_2| = \sqrt{3 + 1} = 2$, $\arg (z_2) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$. Hence $z_2 = 2 \cos \left(-\frac{\pi}{6}\right)$

a. $z_1 z_2 = 2 \sqrt{2} \cos \left(\frac{\pi}{4} + \frac{\pi}{6}\right) = 2 \sqrt{2} \cos \left(\frac{\pi}{12}\right)$

b. $\frac{z_1}{z_2} = \sqrt{2} \cos \left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}} \cos \left(\frac{5\pi}{12}\right)$
c \[ \frac{z_1^3 z_2^3}{i} = -i(z_1z_2)^3 = -i\left(2\sqrt{2}\text{cis}\left(\frac{5\pi}{12}\right)\right)^3 = \text{cis}\left(\frac{\pi}{2}\right)(16\sqrt{2}\text{cis}\left(\frac{5\pi}{4}\right)) = 16\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right) \]

6 First, we construct a director vector \( \vec{v} = B - A = -2 - 4i \). Then, we multiply it by \( i \) to get a rotation of \( \frac{\pi}{2} \) of this vector. Thus, \( \vec{v}_1 = i(-2 - 4i) = 4 - 2i \).

Finally, \( C = B + \vec{v}_1 = 1 - 2i \), \( D = A + \vec{v}_1 = 3 + 2i \)

7 a \[ |w| = \sqrt{4 + 4} = \sqrt{8} \text{ and } \text{arg}(w) = \tan^{-1}(1) = \frac{\pi}{4} \]. Hence, \( w = \sqrt{8}\text{cis}\left(\frac{\pi}{4}\right) \).

b \[ w^4 = \sqrt{8}^4\text{cis}\left(4\times\frac{\pi}{4}\right) = -64, \quad z^6 = \text{cis}\left(6\times\frac{5\pi}{6}\right) = \text{cis}(5\pi) = -1 \]

Hence, \( w^4z^6 = 64 \)

8 We have \( |z_1||z_2| = 2 \left|\frac{z_1}{z_2}\right| = 2 \rightarrow |z_2|^2 = 2 \), so \( |z_2| = 1, \ |z_1| = 2 \)

\[ \theta_1 + \theta_2 = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}, \quad \theta_1 - \theta_2 = \frac{\pi}{2} \rightarrow \theta_1 = \frac{\pi}{2} + \theta_2, \text{ so } \theta_1 = \frac{\pi}{6}, \ \theta_2 = -\frac{\pi}{3} \]

9 \( A_r = |2\text{cis}(60^\circ) + 3\text{cis}(30^\circ)| = \left|2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right| = \left|1 + \frac{3\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right| = 4.84 \)

10 \( I_r = |20\text{cis}(0^\circ) + 10\text{cis}(60^\circ)| = \left|20 + 10\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right| = |25 + 5\sqrt{3}| = \sqrt{5^4 + 5^2 \times 3} = 5\sqrt{28} = 26.5 \text{ amps.} \)

If connected in reverse, \( I_r = \left|20 - 10\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right| = |15 - 5\sqrt{3}| = 10\sqrt{3} = 17.3 \text{ amps} \)

**Exam style questions**

11a The principal axis is \( \frac{5.5 + 1.5}{2} \)

Hence \( p = 3.5 \)

The amplitude is \( \frac{5.5 - 1.5}{2} = 2 \)

Hence \( q = 2 \)

The period is 120

\[ 120 = \frac{360}{r} \]

Hence \( r = 3 \)

So \( y = 3.5 + 2\cos 3x \)
**b** \( 17.2 < x < 62.6 \)

and \( 137.2 < x \leq 180 \)

**12** Area \( OAD = \frac{1}{2} r^2 \theta \)

\[ = \frac{1}{2} \times 10^2 \times \frac{6}{5} \]

\[ = 60 \text{ cm}^2. \]

Area \( OBC = 2 \times 60 = 120 \text{ cm}^2 \)

\[ \frac{1}{2} \times R^2 \times \frac{6}{5} = 120 \]

\[ \Rightarrow R = \sqrt{200} = 10\sqrt{2} \]

Perimeter

\[ = BC + AD + 2 \times BA \]

\[ = 10\sqrt{2} \times \frac{6}{5} + 10 \times \frac{6}{5} + 2(10\sqrt{2} - 10) \]

\[ = 12\sqrt{2} + 12 + 20\sqrt{2} - 20 \]

\[ = 32\sqrt{2} - 8 \text{ cm}. \]

**13 a**

\[ b \] The principal axis is \[ \frac{16 + 4}{2} = 10 \]

Hence \( p = 10 \)

The amplitude is \[ \frac{16 - 4}{2} (= 6) \]

Hence \( q = 6 \)

The period is \( 2 \times 60 = 120 \)
\[ 120 = \frac{360}{r} \]

Hence \( r = 3 \)

So \( y = 10 - 6\sin 3x \)

\[ 14a \quad |100 \text{ cis}(0) + 180 \text{ cis}(90)| = |100 + 180i| = \sqrt{100^2 + 180^2} = 206V \]

\[ b \quad \text{arg}(100 + 180i) = \tan^{-1} \left( \frac{180}{100} \right) = 60.9^\circ \]

\[ 45 + 60.9 = 105.9^\circ \]

\[ 15a \quad z_1z_2 = 4 \text{ cis} \left( -\frac{\pi}{3} \right)3 \text{ cis} \left( \frac{5\pi}{6} \right) = 12 \text{ cis} \left( \frac{5\pi}{6} - \frac{\pi}{3} \right) \]

\[ = 12 \text{ cis} \left( \frac{\pi}{2} \right) \]

\[ = 0 + 12i \]

\[ b \quad \frac{z_1}{z_2} = \frac{4 \text{ cis} \left( \frac{\pi}{3} \right)}{3 \text{ cis} \left( \frac{5\pi}{6} \right)} = \frac{4}{3} \text{ cis} \left( \frac{\pi}{3} - \frac{5\pi}{6} \right) \]

\[ = \frac{4}{3} \text{ cis} \left( -\frac{7\pi}{6} \right) = \frac{4}{3} \text{ cis} \left( \frac{5\pi}{6} \right) \]

So \( \left( \frac{z_1}{z_2} \right)^3 = \left( \frac{4}{3} \text{ cis} \left( \frac{5\pi}{6} \right) \right)^3 = \frac{64}{27} \text{ cis} \left( \frac{15\pi}{6} \right) \]

\[ = \frac{64}{27} \text{ cis} \left( \frac{\pi}{2} \right) \]

\[ = 0 + \frac{64}{27}i \]

\[ c \quad z_1^2 = 16 \text{ cis} \left( -\frac{2\pi}{3} \right) \]

So \( z_1^2 = 16 \text{ cis} \left( \frac{2\pi}{3} \right) \]

\[ = 16 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \]

\[ = 16 \left( -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \]

\[ = -8 + 8\sqrt{3}i \]

\[ 16f(x) + g(x) = \text{Im} \left( e^{\left( \frac{2x + \pi}{3} \right)} + 2e^{\left( \frac{2x - \pi}{4} \right)} \right) = \text{Im} \left( e^{2\pi i} \left( e^{\frac{\pi}{3}i} + 2e^{\frac{-\pi}{4}i} \right) \right) \]
\[ \text{Im}\left(e^{2x} \times 2.977e^{0.872i}\right) = 2.977 \sin(2x + 0.872) \]

Hence \( r = 2.98, \alpha = 0.872 \)